MAT 155B - FALL 12 - EXAMPLES SECTION 11.9

Question. Show that

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

is a solution of $f'' + f = 0$.

Solution. Write

$$f'(x) = \sum_{n=0}^{\infty} \frac{2n(-1)^n x^{2n-1}}{(2n)!},$$

$$f''(x) = \sum_{n=0}^{\infty} \frac{2n(2n-1)(-1)^n x^{2n-2}}{(2n)!} = \sum_{n=0}^{\infty} \frac{2n(2n-1)(-1)^n x^{2(n-1)}}{(2n)!}.$$

Notice that we can begin the above sum at $n = 1$ since the first term, corresponding to $n = 0$, is zero. So

$$f''(x) = \sum_{n=1}^{\infty} \frac{2n(2n-1)(-1)^n x^{2(n-1)}}{(2n)!}.$$

Defining $m = n - 1$, the last sum can be rewritten as

$$\sum_{m=0}^{\infty} \frac{2(m+1)(2(m+1)-1)(-1)^{m+1} x^{2m}}{(2(m+1))!}.$$

(notice that when $n = 1$ we obtain $m = 0$, so this new sum begins at $m = 0$). Now m is just a dummy label, so we can call it n again and we obtain

$$f''(x) = \sum_{n=0}^{\infty} \frac{(2n+2)(2n+1)(-1)^{n+1} x^{2n}}{(2n+2)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!}.$$

This can be written as

$$f''(x) = \sum_{n=0}^{\infty} \frac{(-1)(-1)^n x^{2n}}{(2n)!} = -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Therefore

$$f'' + f = -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 0,$$

showing the result.

Now here is **common question:** "When you change m back to n, then the next n means a different thing than the original n, because, since $m = n - 1$, setting m to n would say $n = n - 1$, and that is like saying $1 = 2$. So, even though the two equations end up looking the same, one is not like the other, so I don't see how this is legitimate".

This usually causes some confusion when seen for the first time, but it is totally legitimate. The point raised is correct, though: the first n and the second n don't mean exactly same thing. But it does not matter, the use of n or m is just a shorthand notation for not writing the series term by term. The series we got with m is:

$$\sum_{m=0}^{\infty} \frac{2(m+1)(2(m+1)-1)(-1)^{m+1}x^{2m}}{(2(m+1))!}.$$

If you expand this series term by term, it looks like:

$$-1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \dots$$

Now, given this series term by term, you don't need to know that it came from a sum with m, n or any other index. You can simply say "let us rewrite it using the summation symbol" and in doing so you can choose the letter you want for indexing it:

$$-1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \dots = -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Another way of thinking is the following. You don't really need to introduce the index m. You could simply get the second derivative:

$$f''(x) = \sum_{n=0}^{\infty} \frac{2n(2n-1)(-1)^n x^{2(n-1)}}{(2n)!},$$

and expand it term by term, obtaining:

$$-1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \dots$$

which is exactly

$$-\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

The step changing to m and then relabeling to n again is just a shortcut to avoid expanding the series term by term.

A final word: the index of summation is really a dummy index. Think of it as the variable of integration on an integral. For example, if you have

$$\int y^3 dy$$

you can rewrite it as $\int x^3 dx$, or $\int u^3 du$..., it does not matter. Even if you got $\int y^3 dy$ after, say, doing some substitution, or integration by parts, or something else, and even though the original integral was an integral with respect to the x variable, once you have $\int y^3 dy$, it is perfect legitimate to rewrite it as $\int x^3 dx$.

URL: `http://www.disconzi.net/Teaching/MAT155B-Fall-12/MAT155B-Fall-12.html`