MAT 155B - FALL 12 - EXAMPLES SECTION 11.10

Question 1: Find the Maclaurin series for $f(x)$.

$$(a) f(x) = \sin(\pi x) \quad (b) f(x) = e^x + e^{2x} \quad (c) f(x) = \frac{x}{\sqrt{4+x^2}}$$

Question 2: Find the Taylor series for $f(x)$ centered at the given value of a.

$$(a) f(x) = \frac{1}{\sqrt{x}},\ a = 9 \quad (b) f(x) = e^x,\ a = 3 \quad (c) f(x) = x^4 - 3x^2 + 1,\ a = 1$$

Question 3: Use series to evaluate the limit

$$\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$$

Question 4: Find the sum of the series.

$$(a) \sum_{n=0}^{\infty} \frac{5^{4n}}{n!} \quad (b) \sum_{n=2}^{\infty} \frac{5^{4n}}{n!} \quad (c) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{n5^n} \quad (d) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!}$$

Solutions.

For all questions, recall the formula for the Taylor series centered at a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{1}$$

The Maclaurin series is simply the Taylor series with $a = 0$.

(1a) Since we want the Maclaurin series, we will use formula (1) with $a = 0$. For $f(x) = \sin(\pi x)$, compute (you need to remember the basic values of sine and cosine, e.g., $\sin \pi$ etc).

$$f(0) = \sin(0) = 0$$
$$f'(x) = \pi\cos(\pi x) \Rightarrow f'(0) = \pi$$
$$f''(x) = -\pi^2 \sin(\pi x) \Rightarrow f''(0) = 0$$
$$f'''(x) = -\pi^3 \cos(\pi x) \Rightarrow f'''(0) = -\pi^3$$
$$f''''(x) = \pi^4 \sin(\pi x) \Rightarrow f''''(0) = 0$$

So after four derivatives, we get back the function $\sin(\pi x)$, multiplied by a power of π. And every time a derivative is taken, there is an extra factor of π popping out. Following the pattern, we see that the n^{th} derivative is zero if n is even, and $\pm\pi^n$ is n is odd, with the sign being "plus" if n is $1, 5, 9, \ldots$ and "minus" if n is $3, 7, 11, \ldots$. We can summarize this by the formulas

$$f^{(2n)}(0) = 0, \quad \text{for all } n$$
$$f^{(2n+1)}(0) = (-1)^n \pi^{2n+1}, \quad \text{for all } n$$

Using formula (1) we then find

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} x^{2n+1}$$

The ratio test shows that the radius of convergence is $R = \infty$.

There is quicker way of doing this problem. The Maclaurin series for $\sin x$ is

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

We can then simply replace x by πx in the above formula to find the answer.

Remark: In most problems, it is easier to use the formulas that we already know for the Taylor and Maclaurin series of commonly used functions, and therefore for the remaining problems we will make use of this shortcut whenever possible. Nevertheless, it is very important that you know how to use formula (1).

(1b) Recall

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

with radius of convergence $R = \infty$. Replacing x by $2x$ we find

$$e^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$$

Adding we find

$$e^x + e^{2x} = \sum_{n=0}^{\infty} \frac{2^n + 1}{n!} x^n$$

with $R = \infty$.

(1c) Notice that

$$f(x) = \frac{x}{\sqrt{4+x^2}} = x(4+x^2)^{-\frac{1}{2}} = \frac{x}{2}\left(1 + \left(\frac{x}{2}\right)^2\right)^{-\frac{1}{2}}$$

Recall the binomial series

$$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n$$

with radius of convergence $R = 1$. In our case $k = -\frac{1}{2}$, and instead of x we have $\left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$, therefore

$$(1+x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})((-\frac{1}{2})-1)((-\frac{1}{2})-2)\cdots((-\frac{1}{2})-n+1)}{n!} \left(\frac{x^2}{4}\right)^n$$

Notice that

$$((-\frac{1}{2}) - 1) = \frac{-1-2}{2} = -\frac{3}{2}$$
$$((-\frac{1}{2}) - 2) = \frac{-1-4}{2} = -\frac{5}{2}$$
$$((-\frac{1}{2}) - n + 1) = \frac{-1-2n+2}{2} = -\frac{2n-1}{2}$$

so

$$(-\frac{1}{2})((-\frac{1}{2}) - 1)((-\frac{1}{2}) - 2)\cdots((-\frac{1}{2}) - n + 1) = (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n}$$

Using this in the formula for the binomial series, writing $\frac{1}{4^n} = \frac{1}{2^{2n}}$ and multiplying by the extra $\frac{x}{2}$, we find

$$\frac{x}{\sqrt{4+x^2}} = \frac{1}{2}x + \sum_{n=1}^{\infty}(-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n! 2^{3n+1}} x^{2n+1}$$

Since the radius of convergence for the binomial series is one, in our case we have

$$\left|\frac{x^2}{4}\right| < 1$$

and so the radius of convergence is $R = 2$.

(2a) Write

$$\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}} = (x-9+9)^{-\frac{1}{2}} = \left(9(1+\frac{x-9}{9})\right)^{-\frac{1}{2}} = \frac{1}{3}(1+\frac{x-9}{9})^{-\frac{1}{2}}$$

Now simply use the binomial series $(1+x)^{-\frac{1}{2}}$ replacing x by $\frac{x-9}{9}$. We find

$$\frac{1}{\sqrt{x}} = \frac{1}{3} + \sum_{n=1}^{\infty}(-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n! 2^n 3^{2n+1}} (x-9)^n$$

with radius of convergence

$$\left|\frac{x-9}{9}\right| < 1 \Rightarrow |x-9| < 9$$

so $R = 9$.

(2b) Let's use formula (1) with $f(x) = e^x$ and $a = 3$. Since the derivative of e^x is itself, we have $f^{(n)}(x) = e^x$ for any n. Hence

$$f^{(n)}(3) = e^3$$

Therefore

$$e^x = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$$

with radius of convergence $R = \infty$.

(2c) Use formula (1) with $f(x) = x^4 - 3x^2 + 1$ and $a = 1$. Compute

$$f'(x) = 4x^3 - 6x \Rightarrow f'(1) = -2$$
$$f''(x) = 12x^2 - 6 \Rightarrow f''(1) = 6$$
$$f'''(x) = 24x \Rightarrow f'''(1) = 24$$
$$f''''(x) = 24 \Rightarrow f''''(1) = 24$$
$$f^{(n)}(x) = 0 \quad \text{for } n \geq 5$$

Hence

$$x^4 - 3x^2 + 1 = -1 - 2(x-1) + 3(x-1)^2 + 4(x-1)^3 + (x-1)^4$$

Since this is a finite sum, $R = \infty$.

Remark: The Taylor series of a polynomial is always a finite sum, since derivatives eventually vanish, and then $R = \infty$.

(3) Recall the Taylor series for $\sin x$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

So

$$\begin{aligned}
\frac{\sin x - x + \frac{1}{6}x^3}{x^5} &= \frac{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} - x + \frac{1}{6}x^3}{x^5} \\
&= \frac{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \sum_{n=3}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} - x + \frac{1}{6}x^3}{x^5} \\
&= \frac{\frac{1}{120}x^5 + \sum_{n=3}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}{x^5} \\
&= \frac{1}{120} + \sum_{n=3}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1-5}
\end{aligned}$$

Where we used $3! = 6$ and $5! = 120$. Notice that since the sum now starts at $n = 3$, the power on x^{2n+1-5} is always positive, hence $\lim_{x \to 0} x^{2n+1-5} = 0$. Therefore

$$\lim_{x \to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} = \frac{1}{120}$$

(4a) Since

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

we have

$$e^{x^4} = \sum_{n=0}^{\infty} \frac{x^{4n}}{n!}$$

So, plugging $x = 5$ we find

$$\sum_{n=0}^{\infty} \frac{5^{4n}}{n!} = e^{5^4} = e^{625}$$

(4b) This is the same sum as in (4a), except that it starts at $n = 2$, so

$$e^{625} = \sum_{n=0}^{\infty} \frac{5^{4n}}{n!} = 5 + \frac{5^4}{1!} + \sum_{n=2}^{\infty} \frac{5^{4n}}{n!}$$

and then

$$\sum_{n=2}^{\infty} \frac{5^{4n}}{n!} = e^{625} - 5 - 625 = e^{625} - 630$$

(4c) Recall

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

Plugging $x = \frac{3}{5}$ we find

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{n5^n} = \ln(1 + \frac{3}{5}) = \ln(\frac{8}{5})$$

(4d) Since

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

plugging $x = \frac{\pi}{4}$ we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!} = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

URL: `http://www.disconzi.net/Teaching/MAT155B-Fall-12/MAT155B-Fall-12.html`