MAT 155B - FALL 12 - EXAMPLES SECTION 11.1

Question 1. Write a formula for the general term of the sequences below.

(a) $\left\{ -\frac{3}{5}, \frac{4}{7}, -\frac{5}{9}, \frac{6}{11}, \dots \right\}$.

(b) $\left\{ 1, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \dots \right\}$.

Question 2. Determine whether the given sequence converges or diverges.

(a) $\left\{ -2, +2, -2, +2, \dots \right\}$.

(b) $a_n = \frac{1}{2^n}$.

(c) $\left\{ \frac{n!}{n^n} \right\}_{n=1}^{\infty}$.

(d) $a_n = (-1)^n \frac{n^2}{(n+7)^2}$.

SOLUTIONS.

1a. The numerator increases by one, starting at 3, so we can put $n+2$ with n starting at 1. The denominator is always odd, but instead of $2n+1$ we have to use $2n+3$ to start at 5. Finally, $(-1)^n$ gives the alternating signs, so that

$$a_n = \frac{(-1)^n (n+2)}{2n+3},\ n \geq 1.$$

1b. The denominator takes the values 1, $2 = 2 \cdot 1$, $6 = 3 \cdot 2 \cdot 1$, $24 = 4 \cdot 6 = 4 \cdot 3 \cdot 2 \cdot 1$, etc, so we recognize it as $n!$. Since $0! = 1$, we have

$$a_n = \frac{1}{n!},\ n \geq 0.$$

2a. Since $a_n = 2$ for n even and -2 for n odd, a_n does not approach any definite value, hence the sequence diverges.

2b. Compute

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^n} = 0,$$

since $2^n \to \infty$ as $n \to \infty$.

2c. Write

$$0 \leq \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots n}{n \cdot n \cdot n \cdot n \cdots n},$$

where n multiplies itself n times on the denominator. But

$$\frac{2 \cdot 3 \cdot 4 \cdots n}{n \cdot n \cdot n \cdots n} \leq 1,$$

where n multiplies itself $n-1$ times on the denominator. Therefore

$$0 \leq \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots n}{n \cdot n \cdot n \cdot n \cdots n} = \frac{1}{n}\Big(\frac{2 \cdot 3 \cdot 4 \cdots n}{n \cdot n \cdot n \cdots n}\Big) \leq \frac{1}{n},$$

i.e.,

$$0 \leq \frac{n!}{n^n} \leq \frac{1}{n}.$$

Since $\frac{1}{n} \to \infty$ as $n \to \infty$, we conclude by the squeeze theorem that

$$\lim_{n \to \infty} \frac{n!}{n^n} = 0.$$

2d. Notice that

$$\lim_{n \to \infty} \frac{n^2}{(n+7)^2} = 1.$$

Hence, a_n approaches 1 for even values of n, and -1 for odd values of n, and we conclude that the sequence diverges.