MAT 155B - FALL 12 - EXAMPLES OF SECTIONS 10.1 AND 10.2

Question 1. Eliminate the parameter in $x = \tan^2 t$, $y = \sec t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$, to find the Cartesian equation of the curve.

Question 2. For which values of t is the curve $x = \cos 2t$, $y = \sin t$, $0 < t < \pi$, concave upward?

Question 3. Set up an integral that represents the length of the curve $x = t + \sqrt{t}$, $y = t - \sqrt{t}$, $0 \leq t \leq 1$.

Solutions.

1. Since

$$1 + \tan^2 t = \sec^2 t,$$

we obtain

$$1 + x = y^2 \Rightarrow x = y^2 - 1.$$

For $-\frac{\pi}{2} < t < \frac{\pi}{2}$ we have $x \geq 0$ and $y \geq 1$. Hence the curve is the portion of the parabola $x = y^2 - 1$ in the first quadrant.

2. We have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sin t}{-2\sin 2t} = \frac{\sin t}{4\sin t \cos t} = \frac{1}{4\cos t} = \frac{1}{4}\sec t,$$

where we used $\sin 2t = 2\sin t \cos t$. Computing the second derivative,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{1}{4}\sec t\right)}{2\sin 2t} = \frac{\frac{1}{4}\sec t \tan t}{-4\sin t \cos t} = -\frac{1}{16}\sec^3 t.$$

Recall that a curve is concave up when the second derivative is positive. Hence we need

$$-\frac{1}{16}\sec^3 t > 0 \Rightarrow \sec^3 t < 0 \Rightarrow \sec t = \frac{1}{\cos t} < 0 \Rightarrow \cos t < 0.$$

For $0 < t < \pi$, $\cos t$ is negative when $\frac{\pi}{2} < t < \pi$.

3. Recall that

$$L = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Computing, we find

$$\frac{dx}{dt} = 1 + \frac{1}{2\sqrt{t}},\ \frac{dy}{dt} = 1 - \frac{1}{2\sqrt{t}},$$

and therefore

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(1 + \frac{1}{2\sqrt{t}}\right)^2 + \left(1 - \frac{1}{2\sqrt{t}}\right)^2 = 2 + \frac{1}{2t}.$$

Hence,

$$L = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 \sqrt{2 + \frac{1}{2t}}\, dt.$$

URL: `http://www.disconzi.net/Teaching/MAT155B-Fall-12/MAT155B-Fall-12.html`