VANDERBILT UNIVERSITY
MAT 155B, FALL 12 — PRACTICE TEST 4

Important: The goal of the practice test is to give you an idea of the sort of questions which will be asked in the exam (e.g., finding a Taylor series, computing the sum of a series, etc), and which sections of the textbook will have more emphasis. The practice test *does not* indicate what *type* of series will appear in the test. For example, because in problem 3 below there is no question with $\ln x$, it does not mean that you should not study Taylor and Maclaurin series involving $\ln x$ (in fact, you should know/memorize all the series on table 1, page 786, of the textbook).

Question 1. Find the radius and interval of convergence of the series.

(a). $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2 5^n}$

Solution. Use the ratio test with $a_n = \frac{x^n}{n^2 5^n}$,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^2 5^{n+1}} \frac{n^2 5^n}{x^n} \right| = \frac{|x|}{5} < 1 \Rightarrow |x| < 5.$$

So $R = 5$.

Plugging in $x = -5$ and $x = 5$, we obtain, respectively,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

which are both convergent, so $I = [-5, 5]$.

(b). $\displaystyle\sum_{n=1}^{\infty} n \frac{(x+1)^n}{4^n}$

Solution. Notice that in this problem $a = -1$. Use the ratio test with $a_n = \frac{(x+1)^n}{4^n}$,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)(x+1)^{n+1}}{4^{n+1}} \frac{4^n}{n(x+1)^n} \right| = \frac{|x+1|}{4} < 1 \Rightarrow |x+1| < 4.$$

So $R = 4$. The interval $|x+1| < 4$ is $-5 < x < 3$. When $x = -5$ or $x = 3$, both series diverge by the divergence test, so $I = (-5, 3)$.

(c). $\displaystyle\sum_{n=1}^{\infty} \frac{n!x^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$

Solution. Use the ratio test with $a_n = \frac{n!x^n}{1\cdot3\cdot5\cdots(2n-1)}$,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{1\cdot3\cdot5\cdots(2n-1)(2n+1)}\frac{1\cdot3\cdot5\cdots(2n-1)}{n!x^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(n+1)x}{2n+1}\right| = \frac{|x|}{2} < 1 \Rightarrow |x| < 2.$$

So $R = 2$. When $x = \pm 2$:

$$|a_n| = \frac{n!2^n}{1\cdot3\cdot5\cdots(2n-1)} = \frac{1\cdot2\cdot3\cdots n\cdot 2^n}{1\cdot3\cdot5\cdots(2n-1)} = \frac{2\cdot4\cdot6\cdots2n}{1\cdot3\cdot5\cdots(2n-1)} > 1,$$

so the series will diverge by the divergence test. Hence $I = (-2, 2)$.

Question 2. Find the radius of convergence of

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}x^n.$$

Solution. Use the ratio test with $a_n = \frac{(2n)!}{(n!)^2}x^n$,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2(n+1))!x^{n+1}}{((n+1)!)^2}\frac{(n!)^2}{(2n)!x^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)x}{(n+1)(n+1)}\right| = 4|x| < 1 \Rightarrow |x| < \frac{1}{4}.$$

So $R = \frac{1}{4}$.

Question 3. Find the Maclaurin series for $f(x)$ and its radius of convergence.

(a). $f(x) = \dfrac{x^2}{1+x}$

Solution. Since for $|x| < 1$,

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-1)^n x^n,$$

we have

$$\frac{x^2}{1+x} = x^2\sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=0}^{\infty}(-1)^n x^{n+2},$$

with $R = 1$.

(b). $f(x) = 10^x$

Solution. Compute

$$f'(x) = 10^x \ln 10 \Rightarrow f'(0) = \ln 10,$$
$$f''(x) = 10^x (\ln 10)^2 \Rightarrow f''(0) = (\ln 10)^2,$$
$$\dots$$
$$f^{(n)}(x) = 10^x (\ln 10)^n \Rightarrow f^{(n)}(0) = (\ln 10)^n.$$

Hence

$$10^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(\ln 10)^n}{n!} x^n$$

To find the radius of convergence,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(\ln 10)^{n+1} x^{n+1}}{(n+1)!} \frac{n!}{(\ln 10)^n x^n} \right| = \lim_{n\to\infty} \frac{\ln 10 |x|}{n+1} = 0,$$

so $R = \infty$.

Different solution. Since

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n,$$

we have

$$10^x = e^{x \ln 10} = \sum_{n=0}^{\infty} \frac{1}{n!} (x \ln 10)^n = \sum_{n=0}^{\infty} \frac{(\ln 10)^n}{n!} x^n,$$

with $R = \infty$ since the radius of convergence for e^x is ∞.

(c). $f(x) = \cosh x$

Solution. Compute

$$f'(x) = \sinh x \Rightarrow f'(0) = 0,$$
$$f''(x) = \cosh x \Rightarrow f''(0) = 1,$$
$$f'''(x) = \sinh x \Rightarrow f'''(0) = 0,$$
$$f''''(x) = \cosh x \Rightarrow f''''(0) = 1,$$
$$\dots$$
$$f^{(n)}(0) = \begin{cases} 1, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Hence

$$\cosh x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

Using the ratio test we find $R = \infty$.

(d). $f(x) = \sin^2 x$

Solution. Since

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x),$$

and

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

with $R = \infty$, we conclude

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \frac{1}{2} - \frac{1}{2} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!}.$$

with $R = \infty$.

(e). $f(x) = xe^{-x}$

Solution. From

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n,$$

with $R = \infty$, we find

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n,$$

so that

$$xe^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+1},$$

with $R = \infty$.

Question 4. Find the Taylor series for $f(x)$ centered at the given value of a.

(a). $f(x) = \dfrac{1}{x}, \ a = -3$

Solution. Compute

$$\begin{aligned}
&f'(x) = -\frac{1}{x^2} \Rightarrow f'(-3) = -\frac{1}{3^2},\\
&f''(x) = \frac{2}{x^3} \Rightarrow f''(-3) = -\frac{2}{3^3}\\
&f'''(x) = -\frac{6}{x^4} \Rightarrow f'''(-3) = -\frac{6}{3^4}\\
&f''''(x) = \frac{24}{x^5} \Rightarrow f''''(-3) = -\frac{24}{3^5}\\
&\dots\\
&f^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}} \Rightarrow f^{(n)}(-3) = -\frac{n!}{3^{n+1}}.
\end{aligned}$$

Then

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(-3)}{n!}(x+3)^n = -\sum_{n=0}^{\infty} \frac{1}{3^{n+1}}(x+3)^n.$$

For the radius of convergence, use the ratio test to find

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(x+3)^{n+1}}{3^{n+2}} \frac{3^{n+1}}{(x+3)^n}\right|\\
&= \frac{|x+3|}{3} < 1 \Rightarrow |x+3| < 3,
\end{aligned}$$

so $R = 3$.

(b). $f(x) = \sqrt{x},\ a = 16$

Solution. Compute

$$\begin{aligned}
&f'(x) = \frac{1}{2}x^{-\frac{1}{2}} \Rightarrow f'(16) = \frac{1}{2}\frac{1}{4},\\
&f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} \Rightarrow f''(16) = -\frac{1}{2}\frac{1}{4^3},\\
&f'''(x) = \frac{3}{8}x^{-\frac{5}{2}} \Rightarrow f'''(16) = \frac{3}{8}\frac{1}{4^5},\\
&f''''(x) = -\frac{15}{16}x^{-\frac{7}{2}} \Rightarrow f''''(16) = -\frac{15}{16}\frac{1}{4^7},\\
&\dots\\
&f^{(n)}(16) = (-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n 4^{2n-1}} = (-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^{5n-2}},\ n \geq 2.
\end{aligned}$$

Then

$$\sqrt{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(16)}{n!}(x-16)^n = 4 + \frac{1}{8}(x-16) + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^{5n-2}n!}(x-16)^n.$$

For the radius of convergence,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1\cdot 3\cdot 5\cdots(2n-1)(x-16)^{n+1}}{2^{5n+3}(n+1)!}\,\frac{2^{5n-2}n!}{1\cdot 3\cdot 5\cdots(2n-3)(x-16)^{n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(2n-1)(x-16)}{2^5(n+1)}\right| = \frac{|x-16|}{16} < 1 \Rightarrow |x-16| < 16,$$

so $R = 16$.

(c). $f(x) = \cos x,\ a = \pi$

Solution. Write

$$\cos x = \cos((x-\pi)+\pi) = \cos(x-\pi)\cos\pi - \sin(x-\pi)\sin\pi = -\cos(x-\pi).$$

Since

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

with $R = \infty$, we conclude that

$$\cos x = -\sum_{n=0}^{\infty}\frac{(-1)^n (x-\pi)^{2n}}{(2n)!},$$

with $R = \infty$.

Question 5. Evaluate the integral as an infinite series.

(a). $\int \arctan(x^2)\,dx$

Solution. Since

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1},$$

with $R = 1$, we have

$$\arctan x^2 = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{2n+1},$$

so

$$\int \arctan x^2\,dx = \int\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{2n+1}\,dx$$

$$= \sum_{n=0}^{\infty}(-1)^n\int\frac{x^{4n+2}}{2n+1}\,dx$$

$$= C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+3}}{(2n+1)(4n+3)},$$

with $R = 1$.

(b). $\displaystyle\int \frac{e^x - 1}{x}\, dx$

Solution. Since

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n,$$

with $R = \infty$, we have

$$\frac{e^x - 1}{x} = \sum_{n=1}^{\infty} \frac{1}{n!} x^{n-1},$$

so that

$$\int \frac{e^x - 1}{x}\, dx = C + \sum_{n=1}^{\infty} \frac{1}{n\, n!} x^n.$$

(c). $\displaystyle\int x \cos(x^3)\, dx$

Solution. Since

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

with $R = \infty$, we conclude that

$$x \cos x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(2n)!},$$

and therefore

$$\int x \cos x^3\, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(6n+2)(2n)!},$$

with $R = \infty$.

Question 6. Use series to approximate the definite integral to within the indicated accuracy.

(a). $\displaystyle\int_0^1 \sin(x^4)\, dx,$ four decimal places

Solution. Since

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

with $R = \infty$, we have

$$\sin x^4 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+4}}{(2n+1)!},$$

so that

$$\int \sin x^4\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+5}}{(8n+5)(2n+1)!},$$

and thus

$$\int_0^1 \sin x^4\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(8n+5)(2n+1)!}.$$

For $n = 3$, the general term is $\frac{1}{29\cdot 7!} \leq 10^{-6} < 10^{-4}$, so, evoking remainder estimates for alternating series, we can sum only up to $n = 2$:

$$\int_0^1 \sin x^4\,dx \approx \sum_{n=0}^{2} \frac{(-1)^n}{(8n+5)(2n+1)!} \approx 0.1876.$$

(b). $\displaystyle\int_0^1 e^{-x^2}\,dx,$ three decimal places

Solution. Since

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n,$$

with $R = \infty$, we have

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n},$$

so

$$\int e^{-x^2}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} x^{2n+1}$$

and therefore

$$\int_0^1 e^{-x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}.$$

Because

$$\frac{1}{(2\cdot 5+1)5!} = \frac{1}{1320} < 10^{-3},$$

we can, evoking remainder estimates for alternating series, we can sum only up to $n = 4$:

$$\int_0^1 e^{-x^2}\,dx \approx \sum_{n=0}^{4} \frac{(-1)^n}{(2n+1)n!}.$$

Question 7. Find the sum of the series.

(a). $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^n}{3^{2n}(2n)!}$

Solution. Notice that

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^n}{3^{2n}(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\sqrt{\pi}}{3} \right)^{2n},$$

which we recognize as the Maclaurin series for $\cos x$ with $x = \frac{\sqrt{\pi}}{3}$, so

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^n}{3^{2n}(2n)!} = \cos \frac{\sqrt{\pi}}{3}.$$

(b). $\frac{1}{1 \cdot 2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7} + \cdots$

Solution. Write

$$\frac{1}{1 \cdot 2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)2^{2n+1}}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)} \left(\frac{1}{2} \right)^{2n+1},$$

which we recognize as the Maclaurin series for $\arctan x$ with $x = \frac{1}{2}$, so

$$\frac{1}{1 \cdot 2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7} + \cdots = \arctan \frac{1}{2}.$$

Question 8. Find the points on the given curve where the tangent line is either horizontal or vertical.

(a). $r = 3 \cos \theta$

Solution. First, let us find a formula for the slope at a point (r, θ). By the chain rule we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}.$$

Since $y = r \sin \theta$, we have

$$\frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta,$$

and similarly, since $x = r \cos \theta$,

$$\frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta.$$

Therefore

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}. \tag{1}$$

From this we easily find that the tangent line is horizontal when

$$-3 \sin^2 \theta + 3 \cos^2 \theta = 3 \cos 2\theta = 0,$$

or $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$. and vertical when

$$-6\sin\theta\cos\theta = -3\sin 2\theta = 0,$$

or $\theta = 0, \frac{\pi}{2}$.

(b). $r = 1 + \cos\theta$

Solution. Use (1) again, so that the tangent line is horizontal when

$$(2\cos\theta - 1)(\cos\theta + 1) = 0,$$

or $\theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$, and vertical when

$$\sin\theta(1 + 2\cos\theta) = 0,$$

or $\theta = 0, \pi, \frac{2\pi}{3}, \frac{4\pi}{3}$.

Question 9. Write an integral representing the area enclosed by one loop of the given curve (do not evaluate the integral).

(a). $r = 4\cos 3\theta$

Solution. The loops close when $r = 4\cos 3\theta = 0$, so the first loop is parametrized by $\theta = -\frac{\pi}{6}$ to $\frac{\pi}{6}$, thus

$$A = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}(4\cos 3\theta)^2\, d\theta.$$

(b). $r^2 = \sin 2\theta$

Solution. The loops close when $r^2 = \sin 2\theta = 0$, so the first loop is parametrized by $\theta = 0$ to $\frac{\pi}{2}$, thus

$$A = \int_{0}^{\frac{\pi}{2}} \frac{1}{2}\sin 2\theta\, d\theta.$$

Question 10. Write an integral representing the length of the given curve (do not evaluate the integral).

(a). $r = \dfrac{1}{\theta}$, $\pi \le \theta \le 2\pi$

Solution. Write

$$L = \int_{\pi}^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{\pi}^{2\pi} \sqrt{\frac{1}{\theta^2} + \left(-\frac{1}{\theta^2}\right)^2}\, d\theta$$
$$= \int_{\pi}^{2\pi} \frac{\sqrt{\theta^2 + 1}}{\theta^2}\, d\theta.$$

(b). $r = \sin^3 \dfrac{\theta}{3},\ 0 \le \theta \le \pi$

Solution. Write

$$L = \int_0^\pi \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^\pi \sqrt{\sin^6 \frac{\theta}{3} + \sin^4 \frac{\theta}{3} \cos^2 \frac{\theta}{3}}\, d\theta.$$