MAT 155B - FALL 12 — SECTIONS 04 AND 13
SOLUTIONS TO THE PRACTICE TEST 3

Question 1. Solve the differential equation

$$y' = 2y(y-2).$$

Write y' as $\frac{dy}{dx}$, move all terms involving y to left hand side and all terms involving x to the right hand side, and integrate to get

$$\int \frac{dy}{y(y-2)} = \int 2\,dx.$$

To compute the y-integral, use partial fractions to get

$$\frac{1}{y(y-2)} = \frac{1}{2}\Big(\frac{1}{y-2} - \frac{1}{y}\Big), \tag{1}$$

so that

$$\ln\left|\frac{y-2}{y}\right| = 4x + C \Longrightarrow \frac{y-2}{y} = Ce^{4x}. \tag{2}$$

Solving for y we find

$$y = \frac{2}{(1 - Ce^{4x})}.$$

In doing the integral, we assumed that $y \neq 0$ and $y \neq 2$. $y = 2$ is also a solution, but this is already included in the above expression (when $C = 0$). $y = 0$ is a solutions as well, as we can verify by plugging it into the equation.

Question 2. What are the constant solutions of the differential equation

$$y' - (y^2 + y^3)\arctan(y + \pi) = 0\,?$$

Solution.

The constant solutions are those such that $y' = 0$. Writing

$$y' = (y^2 + y^3)\arctan(y + \pi),$$

we see that the constant solutions are found by solving

$$(y^2 + y^3)\arctan(y + \pi) = y^2(1 + y)\arctan(y + \pi) = 0.$$

We then find $y = 0$, $y = -1$ and $y = -\pi$.

Question 3. Show that the function $y(x) = xe^{-2x}$ is a solution of the initial value problem

$$\begin{cases} y'' + 4y' + 4y = 0, \\ y(0) = 0,\ y'(0) = 1. \end{cases}$$

Solution. Compute the derivatives to find

$$y' = e^{-2x} - 2xe^{-2x},$$
$$y'' = -4e^{-2x} + 4xe^{-2x}.$$

Therefore

$$\begin{aligned} y'' + 4y' + 4y &= (-4e^{-2x} + 4xe^{-2x}) + 4(e^{-2x} - 2xe^{-2x}) + 4xe^{-2x} \\ &= (-4+4)e^{-2x} + (4-8+4)xe^{-2x} \\ &= 0, \end{aligned}$$

verifying the differential equation. For the initial conditions, plug in zero to get

$$\begin{aligned} y'(0) &= 1, \\ y(0) &= 0. \end{aligned}$$

verifying the initial conditions as well.

Question 4. Determine whether each of the following sequences converges or diverges. You do not have to determine the limit if the sequence converges.

(a) $a_n = \dfrac{(-1)^n n^4}{3n^4+1}$

(b) $a_n = 1 + \dfrac{\sin\frac{n\pi}{2}\ln n}{n}$

(c) $a_1 = 1$, $a_{n+1} = \dfrac{a_n + 9}{2}$

(d) $a_n = \cos\dfrac{(2n+1)\pi}{2}$

Solution.

(a). Write

$$a_n = \frac{(-1)^n n^4}{3n^4+1} = \frac{(-1)^n n^4/n^4}{3n^4/n^4 + 1/n^4} = (-1)^n\left(\frac{1}{3+\frac{1}{n^4}}\right).$$

As $n \to \infty$, the term in parenthesis approaches $\frac{1}{3}$, whereas the entire expression keeps jumping from near $\frac{1}{3}$ (n even) to near $-\frac{1}{3}$ (n odd). Therefore the sequence diverges.

(b). Since $\frac{\ln}{n} \to 0$ (use L'Hospital), and $|\sin\frac{n\pi}{2}| \leq 1$, we have that

$$\left|\frac{\sin\frac{n\pi}{2}\ln n}{n}\right| \to 0.$$

Recall that if $|a_n| \to 0$ then $a_n \to 0$, hence

$$\frac{\sin\frac{n\pi}{2}\ln n}{n} \to 0,$$

and we conclude that the sequence converges.

(c). Since $a_1 \leq 9$, we can argue inductively: if $a_n \leq 9$ then

$$a_{n+1} = \frac{a_n+9}{2} \leq \frac{9+9}{2} = 9,$$

and therefore we conclude that $a_n \leq 9$ for all n. Since $a_n \geq 0$, this sequence is bounded. The sequence is also increasing since

$$a_{n+1} = \frac{a_n+9}{2} \geq \frac{a_n+a_n}{2} = a_n.$$

Therefore it converges by the monotonic sequence theorem.

(d). $\frac{(2n+1)\pi}{2}$ is always an odd multiple of $\frac{\pi}{2}$, hence $a_n = \cos\frac{(2n+1)\pi}{2} = 0$ for every n, and the sequence converges.

Question 4. Find the limit of the sequences below.

(a) $a_n = \dfrac{n}{1+\sqrt{4n^2+1}}$

(b) $a_n = \dfrac{5^n}{n!}$

(c) $a_n = \dfrac{\ln(64n^2+1) - \ln(n^2+n)}{4}$

(d) $a_n = n(1 - e^{\frac{1}{n}})$

Solution.

(a). Write the sequence as

$$a_n = \frac{n}{1+\sqrt{4n^2+1}} = \frac{1}{\frac{1}{n} + \sqrt{4+\frac{1}{n^2}}}.$$

So $a_n \to \frac{1}{0+\sqrt{4+0}} = \frac{1}{2}$.

(b). Notice that

$$0 \leq \frac{5^n}{n!} = \frac{5\cdot5\cdot5\cdot5\cdot5\cdots5}{1\cdot2\cdot3\cdot4\cdot5\cdots n},$$

where 5 multiplies itself n times on the numerator. But

$$\begin{aligned} 0 \leq \frac{5^n}{n!} &= \frac{5\cdot5\cdot5\cdot5\cdot5\cdots5}{1\cdot2\cdot3\cdot4\cdot5\cdots n} \\ &= \Big(\frac{5\cdot5\cdot5\cdot5\cdot5}{1\cdot2\cdot3\cdot4\cdot5}\Big)\Big(\frac{5\cdot5\cdot5\cdots5}{6\cdot7\cdot8\cdot\cdots(n-1)}\Big)\frac{5}{n} \\ &\leq \Big(\frac{5\cdot5\cdot5\cdot5\cdot5}{1\cdot2\cdot3\cdot4\cdot5}\Big).\frac{5}{n} \end{aligned}$$

Since this last expression goes to zero, we conclude that $a_n \to 0$ by the squeeze theorem.

(c). Write the sequence as

$$\begin{aligned} a_n &= \frac{\ln(64n^2+1) - \ln(n^2+n)}{4} = \frac{1}{4}\ln\frac{64n^2+1}{n^2+n} \\ &= \frac{1}{4}\ln\frac{64+\frac{1}{n^2}}{1+\frac{1}{n}}, \end{aligned}$$

so that

$$\begin{aligned} a_n &\to \frac{1}{4}\ln\frac{64+\frac{1}{\infty}}{1+\frac{1}{\infty}} = \frac{1}{4}\ln\frac{64+0}{1+0} \\ &= \frac{1}{4}\ln 2^6 = \frac{3}{2}\ln 2. \end{aligned}$$

(d). We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n(1-e^{\frac{1}{n}}) = \lim_{n\to\infty} \frac{1-e^{\frac{1}{n}}}{\frac{1}{n}}.$$

Letting $x = \frac{1}{n}$, this is the same as

$$\lim_{x\to 0} \frac{1-e^x}{x} = \frac{0}{0} \overset{L'H}{=} \lim_{x\to 0} \frac{0-e^x}{1} = -1.$$

Question 5. Determine whether each of the following series converges or diverges. You do not have to compute the sum if the series converges.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n - 2^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \tan^2\left(\frac{1}{n}\right)$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{\sqrt{4^n+1}}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^{(-1)^n \sin n} + n^4}$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+4)}$

Solution.

(a). It diverges, since the sequence of partial sums alternates between -1 and 0 indefinitely.

(b). Since $\frac{1}{3^n-2^n} \geq 0$, we can use the limit comparison test. Let us compare with $\frac{1}{3^n}$:

$$\lim_{n\to\infty} \frac{\frac{1}{3^n-2^n}}{\frac{1}{3^n}} = \lim_{n\to\infty} \frac{1}{1-(2/3)^n} = 1 \neq 0.$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{3^n}$$

converges because it is a geometric series with $r = \frac{1}{3}$, we conclude that

$$\sum_{n=1}^{\infty} \frac{1}{3^n - 2^n}$$

converges.

We could also have used the comparison test:

$$0 \leq \frac{1}{3^n - 2^n} = \frac{1}{2^n(\frac{3^n}{2^n} - 1)} \leq \frac{2}{2^n},$$

and

$$\sum_{n=1}^{\infty} \frac{2}{2^n} = 2 \sum_{n=1}^{\infty} \frac{1}{2^n}$$

converges since it is a geometric series with $r = \frac{1}{2}$.

(c). This is a series of positive terms. Let us compare it with

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

which converges. Compute

$$\lim_{n\to\infty} \frac{\tan^2\left(\frac{1}{n}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\sin^2\left(\frac{1}{n}\right)}{\cos^2\left(\frac{1}{n}\right)\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\sin^2\left(\frac{1}{n}\right)}{\frac{1}{n^2}},$$

where we used that $\cos^2\left(\frac{1}{n}\right) \to 1$. Letting $x = \frac{1}{n}$ the limit becomes

$$\lim_{n\to 0} = \frac{\sin^2 x}{x^2} = 1.$$

where in the last step we used L'Hospital. Therefore

$$\sum_{n=1}^{\infty} \tan^2\left(\frac{1}{n}\right)$$

converges by the limit comparison test.

(d). Since

$$\frac{3^n}{\sqrt{4^n+1}} = \frac{(3/2)^n}{\sqrt{1+\frac{1}{4^n}}} \to \infty, \tag{3}$$

the series diverges by the divergence test.

(e). This is a series of positive terms, so we can apply the comparison test. Since the exponential is always positive, we have

$$\frac{n}{e^{(-1)^n \sin n} + n^4} \le \frac{n}{n^4} = \frac{1}{n^3}.$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

is a p-series with $p > 1$, it converges, and therefore

$$\sum_{n=1}^{\infty} \frac{n}{e^{(-1)^n \sin n} + n^4}$$

converges as well.

(f). We want to apply the alternating series test. Write the series as

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+4)} = \sum_{n=1}^{\infty} (-1)^{n-1} b_n,$$

where $b_n = \frac{1}{\ln(n+4)}$. We have that $b_n > 0$ and $b_n \to 0$ as $n \to \infty$. $\ln x$ is an increasing function, hence $\frac{1}{\ln(n+4)}$ is decreasing, and therefore by the alternating series test the series converges.

Question 6. Determine the sum of the following convergent series:

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 9}$$

Solution. Use partial fractions to write

$$\frac{1}{4n^2 - 9} = \frac{1}{6}\Big(\frac{1}{2n-3} - \frac{1}{2n+3}\Big).$$

Now write

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 9} = \frac{1}{6}\sum_{n=1}^{\infty}\Big(\frac{1}{2n-3} - \frac{1}{2n+3}\Big)$$
$$= \frac{1}{6}\sum_{n=1}^{\infty}\Big(\frac{1}{2n-3} - \frac{1}{2(n+3)-3}\Big).$$

Let $a_n = \frac{1}{2n-3}$. Then we get

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 9} = \frac{1}{6}\sum_{n=1}^{\infty}\Big(a_n - a_{n+3}\Big).$$

From this expression, we see that for $n > 3$, the second term in each pair gets canceled by the first term in the pair three steps later. For example, when $n = 4$ the term a_4 cancels with the term a_{1+3} which comes from a_{n+3} when $n = 1$; when $n = 5$ the term a_5 cancels with the term a_{2+3} which comes from a_{n+3} when $n = 2$; and so on. Hence only the terms a_1, a_2 and a_3 are left, so that

$$\frac{1}{6}\sum_{n=1}^{\infty}\Big(a_n - a_{n+3}\Big) = \frac{1}{6}\Big(a_1 + a_2 + a_3\Big)$$
$$= \frac{1}{6}\Big(-1 + 1 + \frac{1}{3}\Big) = \frac{1}{18}.$$

Question 7. Find all values of p for which the following series converges:

$$\sum_{n=1}^{\infty} n^p \sin^2\left(\frac{1}{n}\right)$$

Solution. This is a series of positive terms, so our usual tests can be applied. Let us compare with

$$\sum_{n=1}^{\infty} n^p \left(\frac{1}{n}\right)^2 = \sum_{n=1}^{\infty} \frac{1}{n^{2-p}} \qquad (*)$$

This series converges if $2 - p > 1$, i.e. $p < 1$, and diverge otherwise. Compute

$$\lim_{n\to\infty} \frac{n^p \sin^2\left(\frac{1}{n}\right)}{\frac{1}{n^{2-p}}} = \lim_{n\to\infty} \frac{\sin^2\left(\frac{1}{n}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty}\left(\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}\right)^2.$$

Letting $x = \frac{1}{n}$, this becomes

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 = 1.$$

Since $(*)$ converges for $p < 1$ and diverges otherwise, by the limit comparison test we conclude that

$$\sum_{n=1}^{\infty} n^p \sin^2\left(\frac{1}{n}\right)$$

converges for $p < 1$ and diverges otherwise.

Question 8. According to the poem by Ogden Nash,

Big fleas have little fleas,
Upon their backs to bite 'em,
And little fleas have lesser fleas,
And so, ad infinitum.

Assume each flea has exactly one flea which bites it. If the largest flea weighs 0.03 grams, and each flea is $\frac{1}{10}$ the weight of the flea it bites, what is the total weight of all the fleas?

Solution. The weight of the first flea is $w_1 = 0.03$, the weight of the second is $w_2 = 0.03 \times \frac{1}{10}$, the weight of the third is $w_3 = 0.03 \times \frac{1}{100}$, and so on. So the total weight of all the fleas is

$$\begin{aligned} w &= w_1 + w_2 + w_3 + \cdots = 0.03 + \frac{0.03}{10} + \frac{0.03}{100} + \ldots \\ &= 0.03 \sum_{n=0}^{\infty} \frac{1}{10^n}. \end{aligned}$$

This is a convergent geometric series since $r = \frac{1}{10}$, and therefore

$$w = \frac{0.03}{1 - \frac{1}{10}} = \frac{1}{30} \text{ grams.}$$

URL: http://www.disconzi.net/Teaching/MAT155B-Fall-12/MAT155B-Fall-12.html