**VANDERBILT UNIVERSITY**
**MAT 155B, FALL 12 — SOLUTIONS TO THE PRACTICE FINAL.**

**Important:** These solutions should be used as a guide on how to solve the problems and they do not represent the format in which answers should be given in the test. In many questions below only a brief explanation containing the main ideas is given, while in the exam you are expected to fully justify your answers.

**Question 1.** Find the derivative of $f(x)$.

**(a).** $f(x) = \arctan(\ln x)$

**(b).** $f(x) = \dfrac{e^{2x+\sin x}}{\cos^{-1} x}$

**(c).** $f(x) = \ln(\arcsin(\arctan x))$

**(d).** $f(x) = \tan\left(\dfrac{\cos x}{e^{\arctan x}}\right)$

**Solutions.** Using direct application of the chain and quotient rules, we find

$$\textbf{(a).}\quad \frac{1}{x(1+(\ln x)^2)}$$

$$\textbf{(b).}\quad \frac{(2+\cos x)e^{2x+\sin x}}{\cos^{-1} x} + \frac{e^{2x+\sin x}}{(\cos^{-1} x)^2\sqrt{1-x^2}}$$

$$\textbf{(c).}\quad \frac{1}{\arcsin(\arctan x)(1+x^2)\sqrt{1-(\arctan x)^2}}$$

$$\textbf{(d).}\quad -\sec^2\left(\frac{\cos x}{e^{\arctan x}}\right)\frac{(1+x^2)\sin x + \cos x}{e^{\arctan x}(1+x^2)}$$

**Question 2.** If $f(x) = \tan(\cos^{-1}(\ln x))$, find the values of $x$ for which $f(x)$ is invertible, and compute $(f^{-1})'(x)$.

**Solution.** We want $\cos^{-1}(\ln x) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \cap [0,\pi] = [0, \frac{\pi}{2})$, hence $\ln x \in (0,1]$, so $x \in (1, e]$. Computing,

$$f'(x) = -\frac{\sec^2(\cos^{-1}(\ln x))}{x\sqrt{1-(\ln x)^2}}.$$

Then

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = -\frac{y\sqrt{1-(\ln y)^2}}{\sec^2(\cos^{-1}(\ln y))},$$

where $y = f^{-1}(x)$.

**Question 3.** Compute the following limits.

**(a).** $\displaystyle\lim_{x\to\infty} \frac{\ln(x+7)}{x}$

**Solution.**

$$\lim_{x\to\infty} \frac{\ln(x+7)}{x} = \frac{\infty}{\infty} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{1}{x+7} = 0.$$

**(b).** $\displaystyle\lim_{x\to 4} (x-4)\csc(x^2-16)$

**Solution.**

$$\lim_{x\to 4} (x-4)\csc(x^2-16) = \lim_{x\to 4} \frac{x-4}{\sin(x^2-16)} = \frac{0}{0} \overset{\text{L'H}}{=} \lim_{x\to 4} \frac{1}{2x\cos(x^2-16)} = \frac{1}{8}.$$

**(c).** $\displaystyle\lim_{x\to -\infty} \frac{e^{-x}}{\ln|x|}$

**Solution.**

$$\lim_{x\to -\infty} \frac{e^{-x}}{\ln|x|} = \lim_{x\to +\infty} \frac{e^{x}}{\ln x} = \frac{\infty}{\infty} \overset{\text{L'H}}{=} \lim_{x\to\infty} xe^x = \infty.$$

**(d).** $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2+9x} - x\right)$

**Solution.**

$$\sqrt{x^2+9x} - x = \left(\sqrt{x^2+9x} - x\right)\left(\frac{\sqrt{x^2+9x}+x}{\sqrt{x^2+9x}+x}\right) = \frac{9x}{\sqrt{x^2+9x}+x}$$

$$\lim_{x\to\infty} \left(\sqrt{x^2+9x} - x\right) = \lim_{x\to\infty} \frac{9x}{\sqrt{x^2+9x}+x} = \frac{9}{2}.$$

**(e).** $\displaystyle\lim_{x\to\infty} \frac{x-\sin x}{x+\sin x}$

**Solution.**

$$\lim_{x\to\infty} \frac{x-\sin x}{x+\sin x} = \lim_{x\to\infty} \frac{1-\frac{\sin x}{x}}{1+\frac{\sin x}{x}} = 1.$$

**(f).** $\displaystyle\lim_{x\to\infty} \left(\frac{x+1}{x-2}\right)^{2x-1}$

**Solution.**

$$y = \left(\frac{x+1}{x-2}\right)^{2x-1} \Rightarrow \ln y = (2x-1)\ln\frac{x+1}{x-2}$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} (2x-1)\ln\frac{x+1}{x-2} = \lim_{x\to\infty} \frac{\ln\frac{x+1}{x-2}}{\frac{1}{2x-1}} = \frac{0}{0} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{3}{2}\frac{(2x-1)^2}{(x+1)(x-2)} = 6.$$

So

$$\lim_{x\to\infty} y = \lim_{x\to\infty} \left(\frac{x+1}{x-2}\right)^{2x-1} = e^6.$$

**Question 4.** Evaluate the following integrals.

**(a).** $\displaystyle\int \frac{1}{(2x+1)^{\frac{2}{3}}}\,dx$

**Solution.** Using $u = 2x+1$,

$$\int \frac{1}{(2x+1)^{\frac{2}{3}}}\,dx = \frac{3}{2}(2x+1)^{\frac{1}{3}} + C.$$

**(b).** $\displaystyle\int e^{\sin x}\cos x\,dx$

**Solution.** Using $u = \sin x$,

$$\int e^{\sin x}\cos x\,dx = e^{\sin x} + C.$$

**(c).** $\displaystyle\int \cos^2 x \sin^2 x\,dx$

**Solution.** Using half-angle identities,

$$\begin{aligned}\int \cos^2 x \sin^2 x\,dx &= \int \frac{1+\cos(2x)}{2}\frac{1-\sin(2x)}{2}\,dx = \frac{1}{4}\int (1-\cos^2(2x))\,dx \\ &= \frac{x}{8} - \frac{1}{8}\int \cos(4x)\,dx = \frac{x}{8} - \frac{\sin(4x)}{32} + C.\end{aligned}$$

**(d).** $\displaystyle\int \frac{2x^3+1}{x^2+1}\,dx$

**Solution.** Use long division to get

$$\frac{2x^3+1}{x^2+1} = 2x + \frac{1-2x}{x^2+1},$$

so

$$\begin{aligned}\int \frac{2x^3+1}{x^2+1}\,dx &= x^2 + \int \frac{1}{x^2+1}\,dx - 2\int \frac{x}{x^2+1}\,dx \\ &= x^2 + \arctan x - \ln(x^2+1) + C.\end{aligned}$$

**(e).** $\displaystyle\int \tan^{-1} x\,dx$

**Solution.** Integrate by parts with $u = \tan^{-1} x$ and $dv = dx$, to find

$$\int \tan^{-1} x\,dx = x\tan^{-1} x - \frac{1}{2}\ln(x^2+1) + C.$$

**(f).** $\displaystyle\int xe^x\,dx$

**Solution.** Integrate by parts with $u = x$ and $dv = e^x\,dx$,

$$\int xe^x\,dx = xe^x - e^x + C.$$

**(g).** $\int \cos(3x)e^{-x}\,dx$

**Solution.** Integrate by parts with $u = \cos 3x$ and $dv = e^{-x}\,dx$, so

$$\int \cos(3x)e^{-x}\,dx = -e^{-x}\cos(3x) - 3\int \sin(3x)e^{-x}\,dx.$$

Integrate by parts again with $u = \sin 3x$ and $dv = e^{-x}\,dx$ to get

$$\int \cos(3x)e^{-x}\,dx = -e^{-x}\cos(3x) + 3e^{-x}\sin(3x) - 9\int \cos(3x)e^{-x}\,dx,$$

hence

$$\int \cos(3x)e^{-x}\,dx = \frac{-e^{-x}\cos(3x) + 3e^{-x}\sin(3x)}{10} + C.$$

**(h).** $\int \frac{2x^2}{x^2+1}\,dx$

**Solution.**

$$\frac{2x^2}{x^2+1} = 2 - \frac{2}{x^2+1},$$

so

$$\frac{2x^2}{x^2+1}\,dx = \int 2\,dx - \int \frac{2}{x^2+1}\,dx = 2x - 2\arctan x + C.$$

**(i).** $\int x^2\cos x\,dx$

**Solution.** Integrate by parts with $u = x^2$ and $dv = \cos x\,dx$,

$$\int x^2\cos x\,dx = x^2\sin x - \int x\sin x\,dx.$$

Integrate by parts again with $u = x$, $dv = \sin x\,dx$,

$$\int x^2\cos x\,dx = x^2\sin x + 2x\cos x - 2\sin x + C.$$

**(j).** $\int \cos\sqrt{x}\,dx$

**Solution.** Put $z = \sqrt{x}$, so that $dx = 2z\,dz$ and

$$\int \cos\sqrt{x}\,dx = 2\int z\cos z\,dz.$$

Now integrate by parts with $u = z$, $dv = \cos z\,dz$ to get

$$\int \cos\sqrt{x}\,dx = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C.$$

**(k).** $\int \frac{(\ln x)^3}{x}\,dx$

**Solution.** Put $u = \ln x$, so

$$\int \frac{(\ln x)^3}{x}\,dx = \int u^3\,du = \frac{1}{4}(\ln x)^4 + C.$$

**(l).** $\displaystyle\int \frac{\ln x}{x^2}\,dx$

**Solution.** Use integration by parts with $u = \ln x$ and $dv = \frac{1}{x^2}\,dx$,

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

**(m).** $\displaystyle\int \frac{x}{\sin^2 x}\,dx$

**Solution.** Use integration by parts with $u = x$ and $dv = \frac{1}{\sin^2 x}\,dx$,

$$\int \frac{x}{\sin^2 x}\,dx = -x\cot x + \int \cot x\,dx = -x\cot x + \ln|\sin x| + C.$$

**(n).** $\displaystyle\int \frac{x-1}{x^2+3x+2}\,dx$

**Solution.** Use partial fractions to find

$$\frac{x-1}{x^2+3x+2} = -\frac{2}{x+1} + \frac{3}{x+2}, \tag{1}$$

so

$$\int \frac{x-1}{x^2+3x+2}\,dx = -2\ln|x+1| + 3\ln|x+2| + C. \tag{2}$$

**(o).** $\displaystyle\int \cos^3 x \sin^2 x\,dx$

**Solution.** Write

$$\int \cos^3 x \sin^2 x\,dx = \int (1-\sin^2 x)\sin^2 \cos x\,dx.$$

Making the substitution $u = \sin x$ we find,

$$\int \cos^3 x \sin^2 x\,dx = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$$

**(p).** $\displaystyle\int \frac{3x}{x^3-3x-2}\,dx$

**Solution.** Notice that

$$x^3 - 3x - 2 = x^3 - 2x^2 + 2x^2 - 4x + x - 2 = (x^2+2x+1)(x-2) = (x+1)^2(x-2).$$

Write

$$\frac{3x}{x^3-3x-2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}.$$

Writing a system as usual, we find $A = -\frac{2}{3}$, $B = 1$, $C = \frac{2}{3}$. Then

$$\int \frac{3x}{x^3 - 3x - 2}\, dx = -\frac{2}{3} \ln|x+1| - \frac{1}{x+1} + \frac{2}{3} \ln|x-2| + C.$$

**Question 4.** Determine whether the integrals below are improper of type I, improper of type II, or neither. When they are improper, determine whether they are convergent or divergent, evaluating them when they are convergent.

**(a).** $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\, dx$

**Solution.** Improper of type II and convergent because

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x}\Big|_0^1 = 2.$$

**(b).** $\displaystyle\int_0^\infty e^{-x}\, dx$

**Solution.** Improper of type I and convergent because

$$\int_0^\infty e^{-x}\, dx = -e^{-x}\Big|_0^\infty = 1.$$

**(c).** $\displaystyle\int_0^2 \frac{1}{x-e}\, dx$

**Solution.** None because $x - e \neq 0$ for $x \in [0, 2]$, so

$$\int_0^2 \frac{1}{x-e}\, dx = \ln|x-e|\Big|_0^2 = \ln|2-e| - 1.$$

**(d).** $\displaystyle\int_0^3 \frac{1}{x\sqrt{x}}\, dx$

**Solution.** Improper of type II and divergent because

$$\int_0^3 \frac{1}{x\sqrt{x}} = \lim_{t\to 0^+} \frac{-2}{\sqrt{x}}\Big|_t^3 = -\frac{2}{\sqrt{3}} + 2\lim_{t\to 0^+} \frac{1}{\sqrt{t}} = \infty.$$

**(e).** $\displaystyle\int_{-\infty}^\infty xe^{-x^2}\, dx$

**Solution.** Improper of type I and convergent because letting $u = x^2$ we find

$$\int_0^\infty xe^{-x^2}\, dx = -\frac{1}{2}e^{-x}\Big|_0^\infty = \frac{1}{2},$$

and

$$\int_{-\infty}^0 xe^{-x^2}\, dx = -\frac{1}{2}.$$

So

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = 0.$$

**(f).** $\displaystyle\int_0^1 \frac{1}{4x-1}\,dx$

**Solution.** Improper of type II and divergent because

$$\int_0^1 \frac{1}{4x-1}\,dx = \int_0^{\frac{1}{4}} \frac{1}{4x-1}\,dx + \int_{\frac{1}{4}}^1 \frac{1}{4x-1}\,dx$$

and

$$\int_0^{\frac{1}{4}} \frac{1}{4x-1}\,dx = \frac{1}{4}\lim_{t\to\frac{1}{4}^-} \ln|4t-1| = -\infty.$$

**(g).** $\displaystyle\int_{-5}^5 \frac{1}{x-\pi}\,dx$

**Solution.** Improper of type II and divergent because

$$\int_{-5}^5 \frac{1}{x-\pi}\,dx = \int_{-5}^{\pi} \frac{1}{x-\pi}\,dx + \int_{\pi}^5 \frac{1}{x-\pi}\,dx$$

and both these integrals diverge.

**Question 5.** Let $y = f(x)$, $a \le x \le b$, be a curve on the plane, written in Cartesian coordinates. Its length is given by

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

**(a).** Suppose that the same curve is given in parametric form by $x = f(t)$ and $y = g(t)$, $t_0 \le t \le t_1$. Show that its length can then be computed by

$$L = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt.$$

What is the relation between $a$, $b$ and $t_0$, $t_1$?

**Solution.** Using the chain rule,

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

So

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\,dx = \sqrt{\frac{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\,dx$$
$$= \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,\frac{1}{\sqrt{\left(\frac{dx}{dt}\right)^2}}\,dx$$
$$= \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,\sqrt{\left(\frac{dt}{dx}\right)^2}\,dx$$
$$= \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,\frac{dt}{dx}\,dx$$
$$= \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt.$$

From this the result follows after noticing that $x(t_0)=a$ and $x(t_1)=b$.

**(b).** Assume now that the same curve is given in polar form by $r=r(\theta)$, $\theta_0\le\theta\le\theta_1$. Show that in this case

$$L=\int_{\theta_0}^{\theta_1}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

What is the relation between $a$, $b$ and $\theta_0$, $\theta_1$?

**Solution.** Since $x=r\cos\theta=r(\theta)\cos\theta$ and $y=r\sin\theta=r(\theta)\sin\theta$, the curve is parametrized by $\theta$ and we can use the previous formula with $\theta$ replacing $t$,

$$L=\int_{\theta_0}^{\theta_1}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta.$$

Computing we find

$$\frac{dx}{d\theta}=\frac{dr}{d\theta}\cos\theta-r\sin\theta,$$

and

$$\frac{dy}{d\theta}=\frac{dr}{d\theta}\sin\theta+r\cos\theta,$$

and using these expressions into the above formula for $L$ we obtain the result.

**Question 6.** A tank with 1000 gallons of water contains 2% of alcohol (per volume). Water with 5% of alcohol enters the tank at a rate 3 gal/min. The mixture is kept homogeneous and is pumped out at the same rate. Find an expression that describes the percentage of alcohol in the tank after $t$ minutes.

**Solution.** Denote the amount of alcohol (in gallons) at time $t$ by $q(t)$. To find an equation for $\frac{dq}{dt}$, notice that

$$\frac{dq}{dt} = \text{ (rate in) } - \text{ (rate out)}$$

The volume remains constant equal to 1000 gallons. We have

$$\text{(rate in) } = 5\% \text{ of } 3\,gal/min = 0.15\,gal/min.$$

The rate out is given by

$$\text{(rate out ) } = \frac{q(t)}{1000} \times 3 = \frac{3q(t)}{1000}\,gal/min$$

Hence

$$\frac{dq}{dt} = 0.15 - \frac{3q}{1000} = \frac{150 - 3q}{1000} = \frac{3}{1000}(50 - q),$$

or

$$\frac{dq}{50 - q} = \frac{3}{1000}\,dt \Rightarrow \int \frac{dq}{50 - q} = \int \frac{3}{1000}\,dt.$$

Computing the integrals we find

$$-\ln|50 - q| = 0.003t + C \Rightarrow q = Ae^{-0.003t} + 50.$$

To find $A$ we need to use the initial condition. At time zero there is 2% of alcohol, which corresponds to 2% of $1000\,gal = 20\,gal$ Hence $q(0) = 20$ and

$$20 = Ae^0 + 50 \Rightarrow A = -30.$$

Therefore

$$q = -30e^{-0.003t} + 50.$$

This is an expression for the amount of alcohol in gallons; the percentage is $q/1000$.

**Question 7.** Determine whether the series below are convergent or divergent.

**(a).** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$

**Solution.** Since $\ln(n+1) \le n+1$ for large $n$,

$$\frac{1}{\ln(n+1)} \ge \frac{1}{n+1},$$

and the series diverges by comparing with the harmonic series.

**(b).** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{6n^2 + n - 1}$

**Solution.** Converges by the limit comparison test with $\frac{1}{n^2}$.

**(c).** $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^n}{\ln(n^2+3)}(\ln n + \ln(n+4))n^{-n}$

**Solution.** Notice that

$$\begin{aligned}\lim_{n\to\infty}\frac{(n+1)^n}{\ln(n^2+3)}(\ln n+\ln(n+4))n^{-n} &= \lim_{n\to\infty}\frac{\ln n+\ln(n+4)}{\ln(n^2+3)}\left(\frac{n+1}{n}\right)^n\\ &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\lim_{n\to\infty}\frac{\ln(n^2+4n)}{\ln(n^2+3)}\\ &= e\lim_{n\to\infty}\frac{\ln(n^2+4n)}{\ln(n^2+3)}.\end{aligned}$$

But

$$\lim_{n\to\infty}\frac{\ln(n^2+4n)}{\ln(n^2+3)}=\frac{\infty}{\infty}\overset{\text{L'H}}{=}\lim_{n\to\infty}\frac{\frac{2n+4}{n^2+4n}}{\frac{2n}{n^2+3}}=\lim_{n\to\infty}\frac{2n+4}{2n}\frac{n^2+3}{n^2+4n}=1.$$

Hence

$$\lim_{n\to\infty}\frac{(n+1)^n}{\ln(n^2+3)}(\ln n+\ln(n+4))n^{-n}=e\neq 0,$$

and the series diverges by the divergence test.

**(d).** $\displaystyle\sum_{n=1}^{\infty}\sin\frac{1}{n}$

**Solution.** Diverges by using the limit comparison test with $\frac{1}{n}$.

**(e).** $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\ln(n+7)}$

**Solution.** Converges by the alternating series test.

**(f).** $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{12^n}$

**Solution.** Diverges by the ratio test.

**(g).** $\displaystyle\sum_{n=1}^{\infty}\frac{n^3}{\sqrt{n^9+12n^3}}$

**Solution.** Converges by the limit comparison test with $\frac{1}{n^{\frac{3}{2}}}$.

**(h).** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3^n-2}$

**Solution.** Converges by the limit comparison test with $\frac{1}{3^n}$.

**(i).** $\displaystyle\sum_{n=1}^{\infty}\tan\frac{(-1)^n}{n}$

**Solution.** Since tan is an odd function, we have

$$\tan\frac{(-1)^n}{n} = (-1)^n \tan\frac{1}{n}.$$

Because tan is an increasing function on $[0,1]$, and $\frac{1}{n}$ is decreasing for all $n \geq 1$, we have that $\tan\frac{1}{n}$ is decreasing, and since $\tan\frac{1}{n} \to 0$ as $n \to \infty$, the series converges by the alternating series test.

**(j).** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{4^n}$

**Solution.** Converges by the ratio test.

**(k).** $\displaystyle\sum_{n=4}^{\infty} \frac{(n+1)}{(n\ln n)^2}$

**Solution.** Using the limit comparison test with $\frac{1}{n(\ln n)^2}$:

$$\lim_{n\to\infty} \frac{\frac{(n+1)}{(n\ln n)^2}}{\frac{1}{n(\ln n)^2}} = \lim_{n\to\infty} \frac{(n+1)n(\ln n)^2}{(n\ln n)^2} = \lim_{n\to\infty} \frac{n(n+1)}{n^2} = 1.$$

But

$$\sum_{n=4}^{\infty} \frac{1}{n(\ln n)^2}$$

converges by the integral test, so the series

$$\sum_{n=4}^{\infty} \frac{(n+1)}{(n\ln n)^2}$$

converges.

**(l).** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^{e^n}}$

**Solution.** Use the ratio test

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{e^{e^{n+1}}}}{\frac{n!}{e^{e^n}}} = \frac{(n+1)!e^{e^n}}{n!e^{e^{n+1}}} = (n+1)e^{e^n - e^{n+1}} = (n+1)e^{e^n(1-e)}.$$

Since $1 - e \approx -1.71\ldots$, we see that

$$e^{e^n(1-e)} \leq e^{-e^n}.$$

But

$$\lim_{n\to\infty} (n+1)e^{-e^n} = \lim_{n\to\infty} \frac{n+1}{e^{e^n}} = \frac{\infty}{\infty} \overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{1}{e^{e^n}e^n} = 0,$$

so

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$$

by the squeeze theorem, and the series converges by the ratio test.

**Question 8.** Evaluate the sum.

**().** $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{5^n n}$

**Solution.**

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{5^n n} = -\ln(1+\frac{1}{5}).$$

**(b).** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+3n+2}$

**Solution.**

$$\sum_{n=1}^{\infty}\frac{1}{n^2+3n+2} = \sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$$
$$= \frac{1}{2} + \text{ terms that cancel out (telescoping) } = \frac{1}{2}.$$

**(c).** $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(4n^2+6n+2)}{(2n+2)!}$

**Solution.**

$$\sum_{n=0}^{\infty}\frac{(-1)^n(4n^2+6n+2)}{(2n+2)!} = \sum_{n=0}^{\infty}\frac{(-1)^n(4n^2+6n+2)}{(2n+2)(2n+1)(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!} = \cos 1.$$

**Question 9.** Find the Maclaurin series for the functions below, along with their radius of convergence.

**(a).** $\ln(1+7x)$

**Solution.**

$$\ln(1+7x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{7^n}{n}x^n,\ R = \frac{1}{7}.$$

**(b).** $x^3\tan^{-1}\left(\dfrac{x^2}{4}\right)$

**Solution.**

$$x^3\tan^{-1}\left(\frac{x^2}{4}\right) = x^3\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{2n+1}(2n+1)}x^{4n+2}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{2n+1}(2n+1)}x^{4n+5},\ R = 2.$$

**(c).** $\dfrac{e^{-x^2}-1}{x}$

**Solution.**

$$\frac{e^{-x^2}-1}{x} = \frac{1}{x}\Big(-1+\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!}\Big) = \frac{1}{x}\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n}}{n!}$$
$$= \frac{1}{x}\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n-1}}{n!},\ R=\infty.$$

**(d).** $\arcsin x$

**Solution.** Recall that

$$\arcsin x = \int \frac{1}{\sqrt{1-x^2}}\,dx.$$

But

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty}\frac{1}{n!}(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)\cdots(-\frac{1}{2}-n+1)(-x^2)^n$$
$$= \sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2}x^{2n},$$

with $R=1$, from which we get

$$\arcsin x = \sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^{2n}(n!)^2(2n+1)}x^{2n+1},\ R=1.$$

**(e).** $(1-3x)^{-5}$

**Solution.**

$$\frac{1}{(1-3x)^5} = \frac{1}{24\cdot 81}\frac{d^4}{dx^4}\frac{1}{1-3x} = \frac{1}{24\cdot 81}\frac{d^4}{dx^4}\sum_{n=0}^{\infty}3^nx^n$$
$$= \frac{1}{24\cdot 81}\sum_{n=0}^{\infty}3^nn(n-1)(n-2)(n-3)x^{n-4}$$
$$= \frac{1}{24\cdot 81}\sum_{n=4}^{\infty}3^nn(n-1)(n-2)(n-3)x^{n-4},\ R=1.$$

**(f).** $\cos(\sqrt{x})-1$

**Solution.**

$$-1+\cos(\sqrt{x}) = -1+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(\sqrt{x})^{2n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}x^n,\ R=\infty$$

**(g).** $f(x)=\begin{cases}\frac{x-\sin x}{x^3} & \text{if } x\neq 0,\\ \frac{1}{6} & \text{if } x=0.\end{cases}$

**Solution.**

$$\frac{x-\sin x}{x^3} = \frac{1}{x^3} - \frac{1}{x^3}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = -\sum_{n=0}^{\infty}(-1)^{n-1}\frac{x^{2n+3}}{(2n+3)!x^3}$$
$$= \frac{1}{6} + \sum_{n=2}^{\infty}(-1)^n\frac{x^{2n}}{(2n+3)!}.$$

Since this expression reduces to $\frac{1}{6}$ when $x=0$, we have

$$f(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n+3)!},\ R=\infty.$$

**(h).** $e^x + 2e^{-x}$

**Solution.**

$$e^x + 2e^{-x} = \sum_{n=0}^{\infty}\frac{1}{n!}x^n + 2\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n$$
$$= \sum_{n=0}^{\infty}\frac{2(-1)^n+1}{n!}x^n,\ R=\infty.$$

**Question 10.** Find the Taylor series for $f(x)$ centered at the given value of $a$. You do not have to find its radius or interval of convergence.

**(a).** $f(x) = x^{\frac{5}{2}},\ a=1$

**Solution.** Compute

$$f'(x) = \frac{5}{2}x^{\frac{3}{2}},$$
$$f''(x) = \frac{5}{2}\frac{3}{2}x^{\frac{1}{2}},$$
$$f'''(x) = \frac{5}{2}\frac{3}{2}\frac{1}{2}x^{-\frac{1}{2}},$$
$$f''''(x) = \frac{5}{2}\frac{3}{2}\frac{1}{2}\left(-\frac{1}{2}\right)x^{-\frac{3}{2}},$$
$$f'''''(x) = \frac{5}{2}\frac{3}{2}\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-\frac{5}{2}},$$
$$f''''''(x) = \frac{5}{2}\frac{3}{2}\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-\frac{7}{2}},$$

so

$$f^{(n)}(x) = \frac{15(-1)^{n+1}}{8}\frac{1\cdot 3\cdots(2n-7)}{2^{n-3}}x^{-\frac{2n-5}{2}},\ n\geq 4.$$

and then

$$f^{(n)}(1) = \frac{15(-1)^{n+1}}{8}\frac{1\cdot 3\cdots(2n-7)}{2^{n-3}},\ n\geq 4.$$

Hence

$$x^{\frac{5}{2}} = 1 + \frac{5}{2}(x-1) + \frac{1}{2!}\frac{5}{2}\frac{3}{2}(x-1)^2 + \frac{1}{3!}\frac{5}{2}\frac{3}{2}\frac{1}{2}(x-1)^3 + \sum_{n=4}^{\infty}\frac{15(-1)^{n+1}}{8}\frac{1\cdot 3\cdots(2n-7)}{2^{n-3}n!}(x-1)^n.$$

**(b).** $f(x) = x^{-5},\ a = 3$

**Solution.** Compute

$$\begin{aligned} f'(x) &= -5x^{-6}, \\ f''(x) &= (-5)(-6)x^{-7}, \\ f'''(x) &= (-5)(-6)(-7)x^{-8} \\ &\cdots \\ f^{(n)}(x) &= (-1)^n 5\cdot 6\cdots(n+4)x^{-(n+5)} = (-1)^n\frac{(n+4)!}{24}x^{-(n+5)} \\ &\Rightarrow f^{(n)}(3) = (-1)^n\frac{(n+4)!}{24\cdot 3^{n+5}}, \end{aligned}$$

so

$$\begin{aligned} x^{-5} &= \sum_{n=0}^{\infty}(-1)^n\frac{(n+4)!}{24\cdot 3^{n+5}n!}(x-3)^n \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{(n+4)(n+3)(n+2)(n+1)}{24\cdot 3^{n+5}}(x-3)^n. \end{aligned}$$

**(c).** $f(x) = \cos x,\ a = \frac{\pi}{4}$

**Solution.** Write

$$\begin{aligned} \cos x &= \cos(x - \frac{\pi}{4} + \frac{\pi}{4}) = \cos(x-\frac{\pi}{4})\cos\frac{\pi}{4} - \sin(x-\frac{\pi}{4})\sin\frac{\pi}{4} \\ &= \frac{\sqrt{2}}{2}\cos(x-\frac{\pi}{4}) - \frac{\sqrt{2}}{2}\sin(x-\frac{\pi}{4}). \end{aligned}$$

Hence

$$\cos x = \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(x-\frac{\pi}{4})^{2n} - \frac{\sqrt{2}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(x-\frac{\pi}{4})^{2n+1}.$$