VANDERBILT UNIVERSITY
MAT 155B, FALL 12 — PRACTICE FINAL.

Question 1. Find the derivative of $f(x)$.

(a). $f(x) = \arctan(\ln x)$

(b). $f(x) = \dfrac{e^{2x+\sin x}}{\cos^{-1} x}$

(c). $f(x) = \ln(\arcsin(\arctan x))$

(d). $f(x) = \tan\left(\dfrac{\cos x}{e^{\arctan x}}\right)$

Question 2. If $f(x) = \tan(\cos^{-1}(\ln x))$, find the values of x for which $f(x)$ is invertible, and compute $(f^{-1})'(x)$.

Question 3. Compute the following limits.

(a). $\displaystyle\lim_{x\to\infty} \frac{\ln(x+7)}{x}$

(b). $\displaystyle\lim_{x\to 4} (x-4)\csc(x^2-16)$

(c). $\displaystyle\lim_{x\to -\infty} \frac{e^{-x}}{\ln|x|}$

(d). $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2+9x} - x\right)$

(e). $\displaystyle\lim_{x\to\infty} \frac{x-\sin x}{x+\sin x}$

(f). $\displaystyle\lim_{x\to\infty} \left(\frac{x+1}{x-2}\right)^{2x-1}$

Question 4. Evaluate the following integrals.

(a). $\displaystyle\int \frac{1}{(2x+1)^{\frac{2}{3}}}\,dx$

(b). $\displaystyle\int e^{\sin x}\cos x\,dx$

(c). $\int \cos^2 x \sin^2 x \, dx$

(d). $\int \frac{2x^3+1}{x^2+1} \, dx$

(e). $\int \tan^{-1} x \, dx$

(f). $\int xe^x \, dx$

(g). $\int \cos(3x)e^{-x} \, dx$

(h). $\int \frac{2x^2}{x^2+1} \, dx$

(i). $\int x^2 \cos x \, dx$

(j). $\int \cos \sqrt{x} \, dx$

(k). $\int \frac{(\ln x)^3}{x} \, dx$

(l). $\int \frac{\ln x}{x^2} \, dx$

(m). $\int \frac{x}{\sin^2 x} \, dx$

(n). $\int \frac{x-1}{x^2+3x+2} \, dx$

(o). $\int \cos^3 x \sin^2 x \, dx$

(p). $\int \frac{3x}{x^3-3x-2} \, dx$

Question 4. Determine whether the integrals below are improper of type I, improper of type II, or neither. When they are improper, determine whether they are convergent or divergent, evaluating them when they are convergent.

(a). $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$

(b). $\displaystyle\int_0^\infty e^{-x}\,dx$

(c). $\displaystyle\int_0^2 \frac{1}{x-e}\,dx$

(d). $\displaystyle\int_0^3 \frac{1}{x\sqrt{x}}\,dx$

(e). $\displaystyle\int_{-\infty}^\infty xe^{-x^2}\,dx$

(f). $\displaystyle\int_0^1 \frac{1}{4x-1}\,dx$

(g). $\displaystyle\int_{-5}^5 \frac{1}{x-\pi}\,dx$

Question 5. Let $y = f(x)$, $a \le x \le b$, be a curve on the plane, written in Cartesian coordinates. Its length is given by

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

(a). Suppose that the same curve is given in parametric form by $x = f(t)$ and $y = g(t)$, $t_0 \le t \le t_1$. Show that its length can then be computed by

$$L = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

What is the relation between a, b and t_0, t_1?

(b). Assume now that the same curve is given in polar form by $r = r(\theta)$, $\theta_0 \le \theta \le \theta_1$. Show that in this case

$$L = \int_{\theta_0}^{\theta_1} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

What is the relation between a, b and θ_0, θ_1?

Question 6. A tank with 1000 gallons of water contains 2% of alcohol (per volume). Water with 5% of alcohol enters the tank at a rate 3 gal/min. The mixture is kept homogeneous and is pumped out at the same rate. Find an expression that describes the percentage of alcohol in the tank after t minutes.

Question 7. Determine whether the series below are convergent or divergent.

(a). $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$

(b). $\displaystyle\sum_{n=1}^{\infty} \frac{1}{6n^2+n-1}$

(c). $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^n}{\ln(n^2+3)}(\ln n + \ln(n+4))n^{-n}$

(d). $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n}$

(e). $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+7)}$

(f). $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{12^n}$

(g). $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{\sqrt{n^9+12n^3}}$

(h). $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n-2}$

(i). $\displaystyle\sum_{n=1}^{\infty} \tan\frac{(-1)^n}{n}$

(j). $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{4^n}$

(k). $\displaystyle\sum_{n=4}^{\infty} \frac{(n+1)}{(n\ln n)^2}$

(l). $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^{e^n}}$

Question 8. Evaluate the sum.

(a). $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{5^n n}$

(b). $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$

(c). $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(4n^2+6n+2)}{(2n+2)!}$

Question 9. Find the Maclaurin series for the functions below, along with their radius of convergence.

(a). $\ln(1+7x)$

(b). $x^3\tan^{-1}\left(\dfrac{x^2}{4}\right)$

(c). $\dfrac{e^{-x^2}-1}{x}$

(d). $\arcsin x$

(e). $(1-3x)^{-5}$

(f). $\cos(\sqrt{x})-1$

(g). $f(x) = \begin{cases} \frac{x-\sin x}{x} & \text{if } x \neq 0, \\ \frac{1}{6} & \text{if } x = 0. \end{cases}$

(h). $e^x + 2e^{-x}$

Question 10. Find the Taylor series for $f(x)$ centered at the given value of a. You do not have to find its radius or interval of convergence.

(a). $f(x) = x^{\frac{5}{2}},\ a = 1$

(b). $f(x) = x^{-5},\ a = 3$

(c). $f(x) = \cos x,\ a = \dfrac{\pi}{4}$