# MAT 155B - FALL 12 – ASSIGNMENT 4 SOLUTIONS

**Due date[1]: Fri, Nov 16th.**

In all questions below, justify your answers to the best of your abilities.

Recall that a series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if the series $\sum_{n=1}^{\infty} |a_n|$ converges. We saw in class that absolute convergence implies convergence, i.e.

$$\text{if } \sum_{n=1}^{\infty} |a_n| \text{ converges, then } \sum_{n=1}^{\infty} a_n \text{ also converges.}$$

We also saw in class that the reciprocal is not true, i.e., we can have a series which is convergent but which does not converge absolutely. The canonical example is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

since this series converges (by the alternating series test) but

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

A series is called **conditionally convergent** if it is convergent but not absolutely convergent.

**Question 1.** Give three examples of conditionally convergent series.

We know since elementary school that the order in which we add terms does not change the resulting sum. Here we shall investigate if the same result is true when we add infinitely many terms, in other words, when we consider series.

In order to address this question, we introduce the following concept: by a **rearrangement** of a series $\sum_{n=1}^{\infty} a_n$ we mean a series obtained by simply changing the order of the terms. For example, given the series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + \dots,$$

a rearrangement could start as follows:

$$a_1 + a_2 + a_5 + a_8 + a_{15} + \dots$$

In this assignment, you will be given directions on how to prove the following surprising result:

**Theorem 1.** *If $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series and $r$ is any real number, then there is a rearrangement of $\sum_{n=1}^{\infty} a_n$ whose sum is $r$.*

[1] You do not need to hand this in during class. Bring it to my office at any time which is convenient for you. If you do not find me there, please slide it under the door.

In other words, theorem 1 says that if $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then, by suitably changing the order in which the terms are added, we can make the rearranged series converge to any number that we want! In order to prove this, start with the following warm-up question.

**Question 2.** The following series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

converges to $\ln 2$, i.e.,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \ln 2.$$

You do not have to show this, i.e., you can assume that the sum of the above series is $\ln 2$. Then, show that there is a rearrangement of the series so that its sum equals to $\frac{3}{2} \ln 2$ (this is actually done in an example on page 761 of the textbook, at the end of section 11.6, so all you have to do here is follow that example, understand it and write the solutions to the problem with your own words).

Given any series $\sum_{n=1}^{\infty} a_n$, we define a series $\sum_{n=1}^{\infty} a_n^+$ whose terms are all the positive terms of $\sum_{n=1}^{\infty} a_n$, and a series $\sum_{n=1}^{\infty} a_n^-$ whose terms are all the negative terms of $\sum_{n=1}^{\infty} a_n$. More precisely, we let

$$a_n^+ = \frac{a_n + |a_n|}{2}, \quad a_n^- = \frac{a_n - |a_n|}{2}.$$

Notice that if $a_n > 0$, then $a_n^+ = a_n$ and $a_n^- = 0$, whereas if $a_n < 0$, then $a_n^- = a_n$ and $a_n^+ = 0$.

**Question 3.** If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, show that both the series $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ are convergent.

**Question 4.** If $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, show that both the series $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ are divergent.

**Question 5.** Fix a number $r$ and let $\sum_{n=1}^{\infty} a_n$ be conditionally convergent. Use the result of question 4 to show that we can take enough positive terms $a_n^+$ so that their sum is greater than $r$. Next, show that we can add just enough negative terms $a_n^-$ so that the cumulative sum is less than $r$. Continue in this manner to obtain theorem 1 (theorem 11.2.6 of the textbook can help you here).

**Question 6.** What does this result tell you about the use of conditionally convergent series in science and engineering?

**Solutions.**

**1.** The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \quad \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sqrt{n}}{n+4}$$

are all convergent by the alternating series test. But

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges because it is a $p$-series with $p = 1$. The series

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$$

diverges by comparing with the harmonic series, since

$$\frac{1}{n} \leq \frac{1}{\ln n}.$$

Finally

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}\sqrt{n}}{n+4} \right| = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+4}$$

diverges as well since

$$\frac{\sqrt{n}}{n+4} \leq \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}},$$

and

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

diverges because it is a $p$-series with $p = \frac{1}{2}$.

**2.** This is done in the textbook, page 761.

**3.** If $a_n$ is non-negative, then $a_n^- = 0$, $a_n = a_n^+$ and then $|a_n| = a_n^+$ (since $a_n^+ \geq 0$, we don't need the absolute value). Similarly, if $a_n$ is non-positive, then $a_n^+ = 0$, $a_n = a_n^-$ and then $|a_n| = -a_n^-$ (in that $a_n^- \leq 0$ and thus $-a_n^- \geq 0$). It follows that

$$0 \leq |a_n^+| \leq |a_n|,$$

and

$$0 \leq |a_n^-| \leq |a_n|,$$

for every $n$. Therefore

$$0 \leq \sum_{n=1}^{\infty} |a_n^+| \leq \sum_{n=1}^{\infty} |a_n|,$$

and

$$0 \leq \sum_{n=1}^{\infty} |a_n^-| \leq \sum_{n=1}^{\infty} |a_n|.$$

Since $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} |a_n| < \infty$, and we conclude that $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ are absolutely convergent as well, and hence convergent.

**4.** Let us show by contradiction that both $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ diverge. Suppose that $\sum_{n=1}^{\infty} a_n^+$ converges. Then so would the series

$$\sum_{n=1}^{\infty} a_n^+ - \frac{1}{2}\sum_{n=1}^{\infty} a_n,$$

in that $\sum_{n=1}^{\infty} a_n$ converges. But

$$\sum_{n=1}^{\infty} a_n^+ - \frac{1}{2}\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(a_n^+ - \frac{1}{2}a_n) = \sum_{n=1}^{\infty}(\frac{a_n + |a_n|}{2} - \frac{1}{2}a_n) = \frac{1}{2}\sum_{n=1}^{\infty}|a_n|,$$

which diverges since by hypothesis $\sum_{n=1}^{\infty} a_n$ is only conditionally convergent.

A similar argument shows that $\sum_{n=1}^{\infty} a_n^-$ cannot converge.

**5.** Notice that since $\sum_{n=1}^{\infty} a_n^+$ diverges and all its terms are positive, we have

$$\sum_{n=1}^{\infty} a_n^+ = +\infty, \tag{1}$$

and since $\sum_{n=1}^{\infty} a_n^-$ diverges and all its terms are negative,

$$\sum_{n=1}^{\infty} a_n^- = -\infty. \tag{2}$$

Fix $r \in \mathbb{R}$. Because of (1), there exists an integer $N_1$ such that

$$\sum_{n=1}^{N_1} a_n^+ > r.$$

Furthermore, we can choose $N_1$ to be the smallest such integer, so that

$$\sum_{n=1}^{N_1} a_n^+ > r \quad \text{but} \quad \sum_{n=1}^{N_1-1} a_n^+ \le r.$$

Analogously, because of (2), we can add $N_2$ terms of the series of negative terms to obtain

$$\sum_{n=1}^{N_1} a_n^+ + \sum_{n=1}^{N_2} a_n^- < r,$$

and with the property that $N_2$ is the smallest such integer, i.e.,

$$\sum_{n=1}^{N_1} a_n^+ + \sum_{n=1}^{N_2-1} a_n^- \ge r.$$

Next, add again just enough positive terms as to have

$$\sum_{n=1}^{N_1} a_n^+ + \sum_{n=1}^{N_2} a_n^- + \sum_{n=N_1+1}^{N_3} a_n^+ > r,$$

and just enough negative terms so that

$$\sum_{n=1}^{N_1} a_n^+ + \sum_{n=1}^{N_2} a_n^- + \sum_{n=N_1+1}^{N_3} a_n^+ + \sum_{N_2+1}^{N_4} a_n^- < r,$$

where again $N_3$ and $N_4$ are the smallest possible integers such that the above inequalities hold.

Proceeding in this way, we have a series whose sequence of partial sums is

$$s_{N_1+N_2+\cdots+N_K} = \sum_{n=1}^{N_1} a_n^+ + \sum_{n=1}^{N_2} a_n^- + \cdots + \sum_{n=N_{K-2}+1}^{N_K} q_n(K),$$

where

$$q_n(K) = \begin{cases} a_n^+ & \text{if } K \text{ is odd,} \\ a_n^- & \text{if } K \text{ is even.} \end{cases}$$

By construction, if $K$ is odd

$$s_{N_1+N_2+\cdots+N_K} > r \quad \text{and} \quad s_{N_1+N_2+\cdots+N_K-1} \leq r,$$

which implies

$$s_{N_1+N_2+\cdots+N_K} - r \leq a_{N_K}^+. \tag{3}$$

Analogously, if $K$ is even

$$s_{N_1+N_2+\cdots+N_K} < r \quad \text{and} \quad s_{N_1+N_2+\cdots+N_K-1} \geq r,$$

from which follows that

$$r - s_{N_1+N_2+\cdots+N_K} \leq -a_{N_K}^- = |a_{N_K}^-|, \tag{4}$$

where the last equality holds because $a_{N_K}^-$ is non-positive. From (3) and (4) we conclude that

$$|s_{N_1+N_2+\cdots+N_K} - r| \leq |a_{N_K}|. \tag{5}$$

By hypothesis, the original series converges, so $a_n \to 0$ as $n \to \infty$. Inequality (5) then gives

$$\lim_{K\to\infty} s_{N_K} = r,$$

i.e., the series

$$\sum_{n=1}^{N_1} a_n^+ + \sum_{n=1}^{N_2} a_n^- + \sum_{n=N_1+1}^{N_3} a_n^+ + \sum_{N_2+1}^{N_4} a_n^- + \cdots \tag{6}$$

converges, and its sum equals to $r$.

In order to finish the proof, we need to verify that (6) is a rearrangement of the original series. This is indeed the case since by construction, for each $n$, $a_n^+$ and $a_n^-$ are equal if and only if $a_n = 0$, and therefore for each $n$ the term $a_n$ appears exactly once in the sum (6).

*URL*: `http://www.disconzi.net/Teaching/MAT155B-Fall-12/MAT155B-Fall-12.html`