MATH 155A FALL 13
PRACTICE FINAL SOLUTIONS.

Question 1. Find an expression for the function whose graph consists of the line segment from the point $(-2,2)$ to the point $(-1,0)$ together with the top half of the circle with center the origin and radius 1.

Solution.

$$f(x) = \begin{cases} -2x-2, & -2 \le x \le -1, \\ \sqrt{1-x^2}, & -1 < x \le 1. \end{cases}$$

Question 2. Compute the following limits.

(a) $\lim\limits_{x \to 0} \cos(x + \sin x)$.

(b) $\lim\limits_{x \to 1^+} \dfrac{x^2-9}{x^2+2x-3}$.

(c) $\lim\limits_{x \to 4^+} \dfrac{4-x}{|4-x|}$.

(d) $\lim\limits_{x \to \infty} \cos\left(\dfrac{1}{x}\right)$.

(e) $\lim\limits_{x \to \infty} (\sqrt{4x^2+3x} - 2x)$.

Solution. (a) 1. (b) $-\infty$. (c) -1. (d) 1. (e) $\frac{3}{4}$.

Question 3. Find the derivative of the following functions. When necessary, use implicit differentiation to find y' in terms of x and y.

(a) $f(x) = \left(x + \dfrac{1}{x}\right)^{\sqrt{7}}$.

(b) $f(x) = \dfrac{\tan x}{1+\cos x}$.

(c) $y = \sec(1+x^2)$.

(d) $f(x) = \dfrac{1}{\sin(x - \sin x)}$.

(e) $y = \sin(\tan\sqrt{1+x^3})$.

(f) $y\sqrt{x^2+y^3} - \cos y = x^3$.

Solution. (a) $\sqrt{7}(x+\frac{1}{x})^{\sqrt{7}-1}(1-\frac{1}{x^2})$. (b) $\frac{(1+\cos x)\sec^2 x+\tan x\sin x}{(1+\cos x)^2}$. (c) $2x\sec(1+x^2)\tan(1+x^2)$. (d) $-\frac{\cos(x-\sin x)(1-\cos x)}{\sin^2(x-\sin x)}$. (e) $\cos(\tan\sqrt{1+x^3})(\sec^2\sqrt{1+x^3})(3x^2/(2\sqrt{1+x^3}))$. (f) $(3x^2-xy(x^2+y^3)^{-\frac{1}{2}})/(\sqrt{x^2+y^3}+\frac{3}{2}y^3(x^2+y^3)^{-\frac{1}{2}}+\sin y)$.

Question 4. The volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height.

(a) Find the rate of change of the volume with respect to the height if the radius is constant.

(b) Find the rate of change of the volume with respect to the radius if the height is constant.

Solution. (a)

$$V = \frac{1}{3}\pi r^2 h \Rightarrow \frac{dV}{dh} = \frac{1}{3}\pi r^2.$$

(a)

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Question 5. The angle of elevation of the sun is decreasing at a rate of 0.25 rad/h. How fast is the shadow cast by a 400-ft-tall building increasing when the angle of elevation of the sun is $\frac{\pi}{6}$?

Solution. Let x be the length of the shadow. We are given $\frac{d\theta}{dt} = -0.25$. $\tan\theta = 400/x$, so $x = 400\cot\theta$, and

$$\frac{dx}{dt} = -400\csc^2\theta\frac{d\theta}{dt}.$$

When $\theta = \frac{\pi}{6}$, we find 400 ft / h.

Question 6. Show that the shortest distance from the point (x_1, y_1) to the straight line $Ax + By + C = 0$ is

$$\frac{|Ax_1 + By_1 + C|}{\sqrt{A^2+B^2}}.$$

Solution. If $B = 0$, the line is vertical and the distance from $x = -\frac{C}{A}$ to (x_1, y_1) is $|x_1 + \frac{C}{A}| = \frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$. Assume $B \neq 0$. The square of the distance from (x_1, y_1) to the line is

$$f(x) = (x-x_1)^2 + (y-y_1)^2,$$

where $y = -\frac{A}{B}x - \frac{C}{B}$. Thus we need to minimize

$$f(x) = (x-x_1)^2 + (-\frac{A}{B}x - \frac{C}{B} - y_1)^2.$$

Computing f', setting it equal to zero and solving for x gives

$$x = \frac{B^2x_1 - ABy_1 - AC}{A^2 + B^2},$$

and we readily check that this is a minimum. Plugging back into f gives

$$f(x) = \frac{(Ax_1 + By_1 + C)^2}{A^2 + B^2},$$

so

$$\sqrt{f(x)} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

Question 7. A metal storage tank with volume V is to be constructed in the shape of a right circular cylinder surmounted by a hemisphere. What dimensions will require the least amount of metal?

Solution. We wish to minimize the area

$$A = 3\pi r^2 + 2\pi rh,$$

where r is the hemisphere radius and h the cylinder height. The volume is

$$V = \pi r^2 h + \frac{2}{3}\pi r^3,$$

that gives

$$h = \frac{V}{\pi r^2} - \frac{2}{3}r.$$

Plugging this back into A produces

$$A = \frac{5}{3}\pi r^2 + \frac{2V}{r}.$$

Minimizing we find

$$r = \sqrt[3]{\frac{3V}{5\pi}},$$

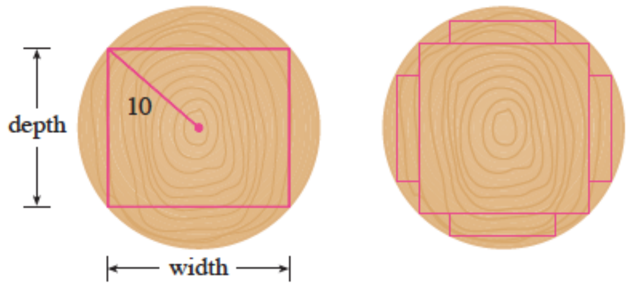
and $h = r$.

Question 8. A rectangular beam will be cut from a cylindrical log of radius 10 inches.

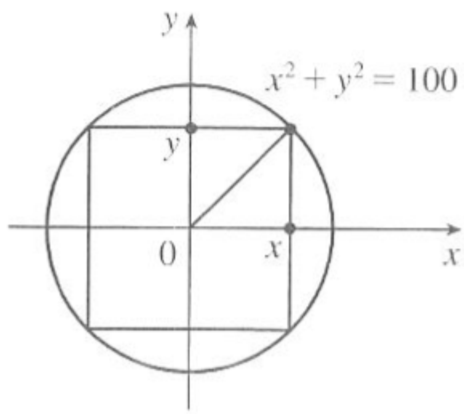
(a) Show that the beam of maximal cross-sectional area is a square.

(b) Four rectangular planks will be cut from the four sections of the log that remain after cutting the square beam. Determine the dimensions of the planks that will have maximal cross-sectional area.

(c) Suppose that the strength of a rectangular beam is proportional to the product of its width and the square of its depth. Find the dimensions of the strongest beam that can be cut from the cylindrical log.



Solution.
(a) Use the picture:



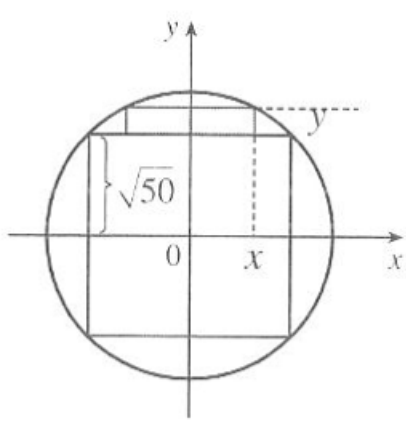
The cross-sections have area

$$A = 4xy = 4x\sqrt{100 - x^2},$$

so

$$\frac{dA}{dx} = \frac{4(-2x^2 + 100)}{(100 - x^2)^{\frac{1}{2}}}.$$

Setting it equal to zero we find $x = \sqrt{50}$. This is a maximum since $A(0) = A(10) = 0$.
(b) Use the picture:



The cross-sectional area of each rectangular plank is

$$A = 2x(y - \sqrt{50}) = 2x(\sqrt{100 - x^2} - \sqrt{50}),$$

so

$$\frac{dA}{dx} = 2(100 - x^2)^{\frac{1}{2}} - 2\sqrt{50} - \frac{2x^2}{(100 - x^2)^{\frac{1}{2}}}.$$

Then

$$(100 - x^2) - \sqrt{50}(100 - x^2)^{\frac{1}{2}} - x^2 = 0 \Rightarrow 100 - 2x^2 = \sqrt{50}(100 - x^2)^{\frac{1}{2}}$$
$$\Rightarrow 4x^4 - 350x^2 + 5000 = 0 \Rightarrow x \approx 8.34 \text{ or } 4.34.$$

But $8.35 > \sqrt{50}$, so $x \approx 4.24$. This gives that each plank should have dimensions $4.24 + 4.24 = 8.48$ by $y - \sqrt{50} = \sqrt{100 - (4.24)^2} - \sqrt{50}$.

(c) From part (a), the width is $2x$ and the depth $2y$, so the strength is

$$S = 8kxy^2 = 8kx(100 - x^2) = 800kx - 8kx^3.$$

Computing S', setting it equal to zero and solving for x we find a maximum when $x = \frac{10}{\sqrt{3}}$.

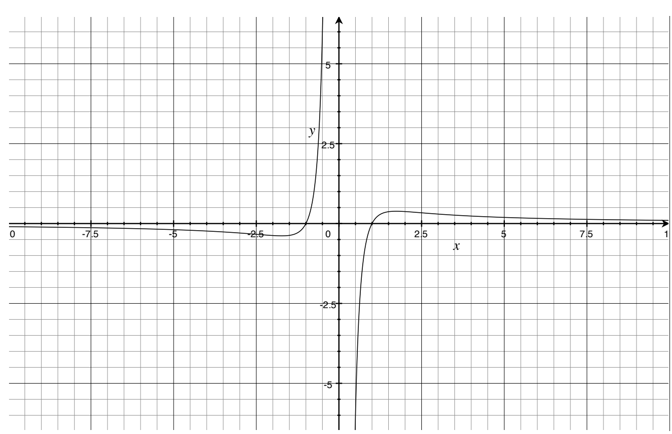
Question 9. Use derivatives to sketch the graph of the following functions.

(a) $f(x) = \dfrac{x^2 - 1}{x^3}$.

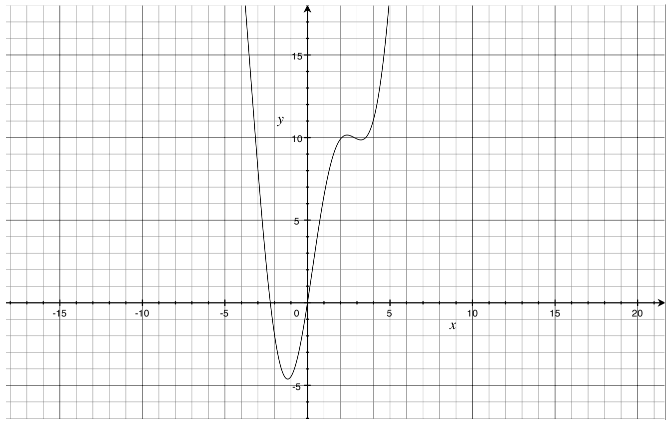
(b) $f(x) = x^2 + 6.5 \sin x$, $-5 \le x \le 5$.

Solution.

(a)



(b)



Question 10. Evaluate the indefinite and definite integrals.

(a) $\int x \sin(x^2)\,dx.$

(b) $\int_0^a x\sqrt{a^2 - x^2}\,dx.$

(c) $\int \sqrt{x} \sin(1 + x^{\frac{3}{2}})\,dx.$

(d) $\int \frac{\cos(\pi/x)}{x^2}\,dx.$

(e) $\int_0^{\frac{\pi}{2}} \cos x \sin(\sin x)\,dx.$

Solution. (a) $-\frac{1}{2}\cos(x^2) + C$. (b) $\frac{1}{3}a^3$, $a > 0$. (c) $-\frac{2}{3}\cos(1 + x^{\frac{3}{2}}) + C$. (d) $-\frac{1}{\pi}\sin\frac{\pi}{x} + C$. (e) $1 - \cos 1$.

Question 11. Use integration to show that the volume of a right cone with a circular base of radius r and height h is $\frac{1}{3}\pi r^2 h$.

Solution. This is like the example of section 5.2 posted on the course webpage. See also the similar example of a pyramid done in class.

Question 12. The base of a solid is the region bounded by the parabolas $y = x^2$ and $y = 2 - x^2$. Find the volume of the solid if the cross-sections perpendicular to the x-axis are squares with one side lying along the base.

Solution.

$$V = \int_{-1}^{1} A(x)\,dx = 2\int_0^1 ((2 - x^2) - x^2)^2\,dx = \frac{64}{15}.$$

Question 13. Find the volumes of the solids obtained by rotating the region bounded by the curves $y = x$ and $y = x^2$ about the following lines.

(a) The x-axis.
(b) The y-axis.
(c) $y = 2$.

Solution.
(a)

$$V = \int_0^1 \pi(x^2 - (x^2)^2)\,dx = \frac{2}{15}\pi.$$

(b)

$$V = \int_0^1 \pi((\sqrt{y})^2 - y^2)\,dy = \frac{1}{6}\pi.$$

(c)

$$V = \int_0^1 \pi((2-x^2)^2 - (2-x)^2)\,dx = \frac{8}{15}\pi.$$

Question 14. A force of 30 N is required to maintain a spring stretched from its natural length of 12 cm to a length of 15 cm. How much work is done in stretching the spring from 12 cm to 20 cm?

Solution. Put $f(x) = kx$, so $30N = k(15-13)cm \Rightarrow k = 1000N/m$. Also, $20cm - 12cm = 0.08m$, thus

$$W = \int_0^{0.08} kx\,dx = 3.2J.$$

URL: `http://www.disconzi.net/Teaching/MAT155A-Fall-13/MAT155A-Fall-13.html`