MATH 155A FALL 13
PRACTICE MIDTERM 4.

Question 1. Find the derivative of the following functions.

(a) $f(x) = \int_0^x \sqrt{t + \sqrt{t}}\, dt$.

Solution. By the FTC,

$$f'(x) = \sqrt{x + \sqrt{x}}.$$

(b) $f(x) = \int_0^{\sqrt{x}} \frac{z^2}{z^4 + 1}\, dz$.

Solution. By the FTC and the chain rule with $u = \sqrt{x}$,

$$f'(x) = \frac{\sqrt{x}}{2(x^2 + 1)}.$$

(c) $f(x) = \int_{\tan x}^{x^2} \frac{1}{2 + t^4}\, dt$.

Solution. Split the integral as

$$\begin{aligned}\int_{\tan x}^{x^2} \frac{1}{2+t^4}\, dt &= \int_{\tan x}^{0} \frac{1}{2+t^4}\, dt + \int_0^{x^2} \frac{1}{2+t^4}\, dt \\ &= -\int_0^{\tan x} \frac{1}{2+t^4}\, dt + \int_0^{x^2} \frac{1}{2+t^4}\, dt.\end{aligned}$$

For the first integral, use the FTC and the chain rule with $u = \tan x$. For the second one, use again the FTC and the chain rule, but now with $u = x^2$. One finds,

$$f'(x) = -\frac{\sec^2 x}{2 + \tan^4 x} + \frac{2x}{2 + x^8}.$$

Question 2. Evaluate the following indefinite integrals.

(a) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx$.

Solution. Substitution with $u = \sqrt{x}$ gives

$$-2\cos\sqrt{x} + C.$$

(b) $\displaystyle\int \frac{dx}{\cos^2 x\sqrt{1+\tan x}}$.

Solution. Substitution with $u = 1 + \tan x$ gives

$$2\sqrt{1+\tan x} + C.$$

(c) $\displaystyle\int x^3\sqrt{x^2+1}\,dx$.

Solution. Substitution with $u = 1 + x^2$, so $x^2 = u - 1$, gives

$$\frac{1}{5}(x^2+1)^{\frac{5}{2}} - \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C.$$

(d) $\displaystyle\int \sin x \cos^4 x\,dx$.

Solution. Substitution with $u = \cos x$ gives

$$-\frac{1}{5}\cos^5 x + C.$$

(e) $\displaystyle\int \Big(\int_0^{\sin x} t\,dt\Big)\cos x \sin x\,dx$.

Solution. Put

$$u = \int_0^{\sin x} t\,dt,$$

so that, by the FTC and the chain rule,

$$du = \sin x \cos x\,dx.$$

Therefore,

$$\begin{aligned}\int \Big(\int_0^{\sin x} t\,dt\Big)\cos x \sin x\,dx &= \int u\,du\\ &= \frac{1}{2}u^2 + C\\ &= \frac{1}{2}\Big(\int_0^{\sin x} t\,dt\Big)^2 + C.\end{aligned}$$

But

$$\int_0^{\sin x} t\,dt = \frac{1}{2}t^2\Big|_0^{\sin x} = \frac{1}{2}\sin^2 x,$$

what finally yields

$$\int \left(\int_0^{\sin x} t\, dt \right) \cos x \sin x\, dx = \frac{1}{8} \sin^4 x + C.$$

Question 3. Evaluate the following definite integrals.

(a) $\displaystyle\int_1^2 \left(\frac{1}{x^2} - \frac{4}{x^3} \right) dx.$

Solution. -1, use power rule.

(b) $\displaystyle\int_1^9 \frac{3x-2}{\sqrt{x}}\, dx.$

Solution. 44, use power rule.

(c) $\displaystyle\int_0^{\frac{3\pi}{2}} |\sin x|\, dx.$

Solution. 3, use

$$\int_0^{\frac{3\pi}{2}} |\sin x|\, dx = \int_0^{\pi} \sin x\, dx + \int_{\pi}^{\frac{3\pi}{2}} (-\sin x)\, dx.$$

Question 4. Find the area enclosed by the given curves.

(a) $x = y^4$, $y = \sqrt{2-x}$, $y = 0$.

Solution. Intersection for $y \geq 0$: $y = 1$.

$$A = \int_0^1 \left((2 - y^2) - y^4 \right) dy = \frac{22}{15}.$$

(b) $y = \cos x$, $y = 1 - \cos x$, $0 \leq x \leq \pi$.

Solution. Intersection at $x = \frac{\pi}{3}$.

$$A = \int_0^{\frac{\pi}{3}} \big(\cos x - (1 - \cos x) \big)\, dx + \int_{\frac{\pi}{3}}^{\pi} \big((1 - \cos x) - \cos x \big)\, dx = 2\sqrt{3} + \frac{\pi}{3}.$$

(c) $y = \dfrac{1}{4}x^2$, $y = 2x^2$, $x + y = 3$, $x \geq 0$.

Solution. Intersection at $x = 1, 2$ for $x \geq 0$.

$$A = \int_0^1 (2x^2 - \frac{1}{4}x^2)\, dx + \int_1^2 \big((-x+3) - \frac{1}{4}x^2 \big)\, dx = \frac{3}{2}.$$

Question 5. Using integration, show that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

Solution. This was shown in class, and it is done on page 353 of the textbook.

Question 6. Using integration, show that the volume of a cone with a circular base of radius r and height h is $\frac{1}{3}\pi r^2 h$.

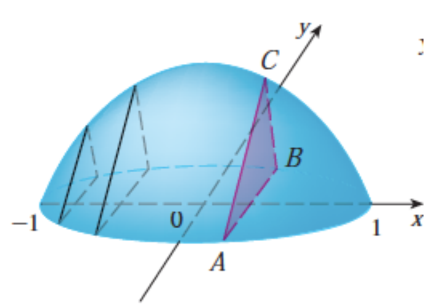
Solution. Cross-sections are disks of radius x, where

$$\frac{x}{y} = \frac{r}{h}.$$

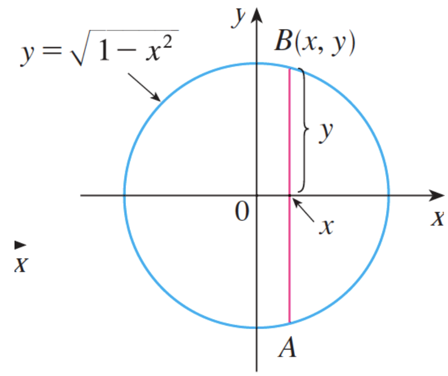
Thus

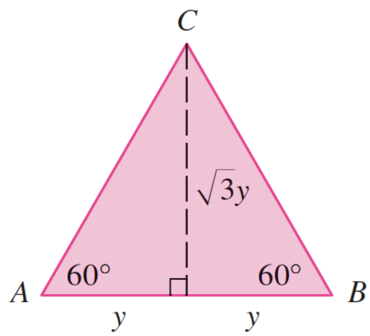
$$\begin{aligned} V &= \int_0^h A(y)\,dy \\ &= \int_0^h \pi x^2\,dy \\ &= \int_0^h \pi\left(\frac{ry}{h}\right)^2\,dy \\ &= \pi\frac{r^2}{h^2}\int_0^h y^2\,dy \\ &= \frac{1}{3}\pi r^2 h. \end{aligned}$$

Question 7. The picture below shows a solid with a circular base of radius 1. Parallel cross sections perpendicular to the base are equilateral triangles. Find the volume of the solid.



Solution. The pictures below depict the base and a typical cross-section.





Since B lies on the circle, we have $y = \sqrt{1-x^2}$, and thus the base of the triangle ABC is $|AB| = 2\sqrt{1-x^2}$. Since the triangle is equilateral, its height is $\sqrt{3}\sqrt{1-x^2}$. The cross-sectional area is then

$$A(x) = \sqrt{3}(1-x^2),$$

and the volume

$$A = \int_{-1}^{1} A(x)\,dx = \frac{4\sqrt{3}}{3}.$$

Question 8. Find the volume of the solid S whose base is a circular disk with radius r and parallel cross sections perpendicular to its base are squares.

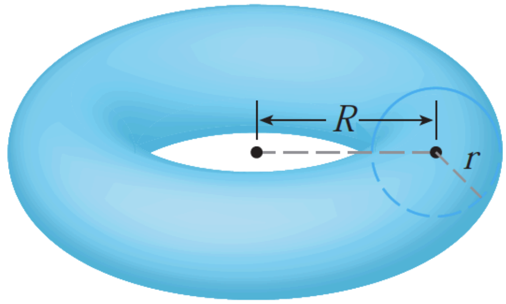
Solution. The area of a cross section is

$$A(x) = (2y)^2 = (r^2 - x^2),$$

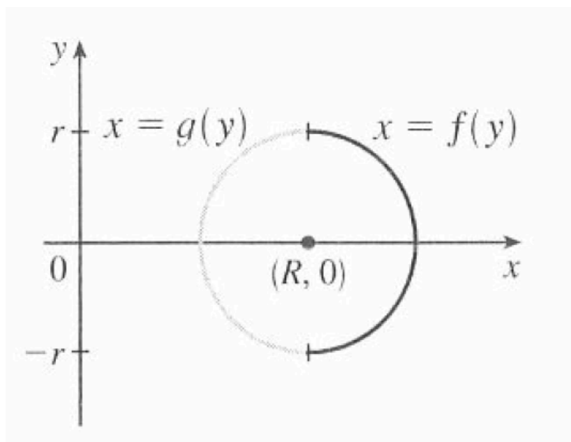
and the volume

$$V = \int_{-r}^{r} A(x)\,dx = \frac{16}{3}r^3.$$

Question 9. Find the volume of a solid torus, the donut-shaped solid shown in the figure below, of radii r and R.



Solution. Consider the picture:



The torus is obtained by rotating the circle $(x-R)^2+y^2=r^2$ about the y-axis. Solving for x, we see that the right half of the circle is given by

$$x = R+\sqrt{r^2-y^2} = f(y),$$

and the left half by

$$x = R-\sqrt{r^2-y^2} = g(y).$$

Notice that $f(y)$ is the outer radius and $g(y)$ the inner radius, thus

$$\begin{aligned}
V &= \pi\int_{-r}^{r}\left((f(y))^2-(g(y))^2\right)dy\\
&= 2\pi\int_0^r\left[\left(R^2+2R\sqrt{r^2-y^2}+r^2-y^2\right)-\left(R^2-2R\sqrt{r^2-y^2}+r^2-y^2\right)\right]dy\\
&= 8\pi R\int_0^r\sqrt{r^2-y^2}\,dy.
\end{aligned}$$

We haven't yet learned how to compute this integral. However, interpreting it in terms of area, we notice that it represents a quarter of the area of a circle of radius r, thus

$$\begin{aligned}
V &= 8\pi R\int_0^r\sqrt{r^2-y^2}\,dy\\
&= 8\pi R\frac{\pi r^2}{4}
\end{aligned}$$

$$= 2\pi^2 r^2 R.$$

Question 10. Set up an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

(a) $y = \sqrt{x-1}$, $y = 0$, $x = 5$ about the $x-$ axis.

Solution.

$$V = \int_1^5 \pi(x-1)\,dx.$$

(b) $y = x$, $y = 0$, $x = 2$, $x = 4$, about $x = 1$.

Solution.

$$V = \pi\int_0^2 \left(4-1)^2 - (2-1)^2\right)dy + \pi\int_2^4 \left((4-1)^2 - (y-1)^2\right)dy.$$

(c) $x^2 + 4y^2 = 4$, about $y = 2$.

Solution.

$$V = \int_{-2}^2 \pi\left\{\left[2 - \left(-\sqrt{1-\frac{x^2}{4}}\right)\right]^2 - \left(2 - \sqrt{1-\frac{x^2}{4}}\right)^2\right\}dx.$$

(d) $x^2 + 4y^2 = 4$, about $x = 2$.

Solution.

$$V = \int_{-1}^1 \pi\left\{\left[2 - \left(-\sqrt{4-4y^2}\right)\right]^2 - \left(2 - \sqrt{4-4y^2}\right)^2\right\}dy.$$

(e) $y^2 - x^2 = 1$, $y = 2$, about the $y-$ axis.

Solution. By cylindrical shells with disks $x = \pm\sqrt{y^2-1}$,

$$V = \pi\int_1^2 \left(\sqrt{y^2-1}\right)^2 dy.$$

(e) $x = (y-3)^2$, $x = 4$, about $x = -1$.

Solution. By cylindrical shells,

$$V = 2\pi\int_1^5 (y-1)\left(4 - (y-3)^2\right)dy.$$

Question 11. Prove the Fundamental Theorem of Calculus.

Solution. This was done in class, and it is done on page 312 of the textbook.

URL: `http://www.disconzi.net/Teaching/MAT155A-Fall-13/MAT155A-Fall-13.html`