MATH 155A FALL 13
SOLUTIONS TO THE PRACTICE MIDTERM 2.

Question 1. Find y'.

(a) $y = \dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt[5]{x^3}}$.

(b) $y = \dfrac{\tan x}{1+\cos x}$.

(c) $y = x^{\sqrt{21+\pi}} \cos x$.

(d) $y = \dfrac{1}{\sin(x - \sin x)}$.

(e) $y = \sin^2(\cos(\sqrt{\sin(\pi x)}))$.

Solution. Direct application of the differentiation rules yields:
(a) $y' = \frac{3}{5x\sqrt[5]{x^3}} - \frac{1}{2x\sqrt{x}}$.
(b)$y' = \frac{(1+\cos x)\sec^2 x + \tan x \sin x}{(1+\cos x)^2}$.
(c) $y' = \sqrt{21+\pi} x^{\sqrt{21+\pi}-1} \cos x - x^{\sqrt{21+\pi}} \sin x$.
(d) $y' = -\frac{\cos(x-\sin x)(1-\cos x)}{\sin^2(x-\sin x)}$.
(e) $y' = \frac{-\pi \sin(\cos\sqrt{\sin(\pi x)}) \cos(\cos\sqrt{\sin(\pi x)}) \sin\sqrt{\sin(\pi x)} \cos(\pi x)}{\sqrt{\sin(\pi x)}}$.

Question 2. Find an equation for the tangent line and normal line to the curve at the given point.
(a) $y = (1+2x)^2,\ (1, 9)$.

(b) $y = \dfrac{\sqrt{x}}{x+1},\ (4, 0.4)$.

(c) $x^2 + 2xy - y^2 + x = 2,\ (1, 2)$.

Solution.
(a) Taking the derivative:

$$y' = 4 + 8x.$$

At $(1, 9)$, $y' = 12$, hence the equation of the tangent line is

$$y - 9 = 12(x - 1).$$

The slope of the normal line is $-\frac{1}{12}$, hence the normal equation is

$$y - 9 = -\frac{1}{12}(x - 1).$$

(b) Similar to (a); the two equations are $y - 0.4 = -0.03(x - 4)$ and $y - 0.4 = \frac{100}{3}(x - 4)$.
(c) Use implicit differentiation to get

$$y' = \frac{-2x - 2y - 1}{2x - 2y}.$$

Using $x = 1$, $y = 2$ we find $y' = \frac{7}{2}$ so that the equation of the tangent and normal lines become, respectively, $y - 2 = \frac{7}{2}(x - 1)$, and $y - 2 = -\frac{2}{7}(x - 1)$.

Question 3. An object of mass m is shot straight upward from the ground with initial velocity v_0. Assuming that there is no air resistance, the height h of the object, measured with respect to the ground level, is given as a function of time, by

$$h(t) = v_0 t - \frac{1}{2} g t^2,$$

where t is the time, and g is the value of the gravitational acceleration. Show that the maximum height of the object is

$$h_{max} = \frac{1}{2}\frac{v_0^2}{g}.$$

Suppose that gravity in the planet Krypton is such that its gravitational acceleration is twice that of Earth's. How much faster would the object have to be shot in order to reach the same height?

Solution. The object attains its maximum height when its velocity vanishes. Computing $v(t) = h'(t)$ and setting it equal to zero gives

$$0 = v_0 - gt \Rightarrow t = \frac{v_0}{g},$$

i.e., the the maximum height is achieved at a time $\frac{v_0}{g}$. Plugging this into $h(t)$ gives the desired value for h_{max}. For the second question, if g doubles, v_0 has to be multiplied by $\sqrt{2}$ for h_{max} to stay the same, hence the answer is $\sqrt{2}v_0$.

Question 4. Find y'' by implicit differentiation.

(a) $x^3 + y^3 = 1$.

(b) $\sqrt{x} + \sqrt{y} = 1$.

Solution.
(a) Differentiating both sides and solving for y' gives

$$y' = -\frac{x^2}{y^2}.$$

Differentiating again

$$y'' = -\frac{2x(y^3 + x^3)}{y^5}.$$

But $y^3 + x^3 = 1$, so this simplifies to

$$y'' = -\frac{2x}{y^5}.$$

(b) Differentiating both sides and solving for y' gives

$$y' = -\frac{\sqrt{y}}{\sqrt{x}}.$$

Differentiating again

$$y'' = \frac{\sqrt{x} + \sqrt{y}}{2x\sqrt{x}}.$$

But $\sqrt{x} + \sqrt{y} = 1$, so this simplifies to

$$y'' = \frac{1}{2x\sqrt{x}}.$$

Question 5. Two sides of a triangle have lengths 12m and 15m. The angle between them is increasing at a rate of 2°/min. How fast is the length of the third side increasing when the angle between the sides of fixed length is 60°?

Solution. Let the angle in question be θ, and x be the size of the third side. The relation between the sides and θ is (notice that this is not a triangle with a 90° angle):

$$x^2 = 12^2 + 15^2 - 2 \times 12 \times 15 \cos\theta = 369 - 360 \cos\theta.$$

Taking $\frac{d}{dt}$:

$$\frac{dx}{dt} = 180 \frac{\sin\theta}{x} \frac{d\theta}{dt}.$$

When $\theta = 60°$ and $\frac{d\theta}{dt} = 2°/min = \frac{\pi}{90} rad/min$:

$$\frac{dx}{dt} = \frac{\sqrt{7}\pi}{21} m/min.$$

Question 6. A balloon is raising at a constant speed of 5 ft/s. A boy is cycling along a straight road at a speed of 15 ft/s. When he passes under the balloon, it is 45 ft above him. How fast is the distance between the boy and the balloon increasing 3 seconds later?

Solution. Let z be their distance, h the balloon's height, and x the distance from the boy to the point directly under the balloon. Then $z^2 = x^2 + h^2$. Taking $\frac{d}{dt}$ and using $\frac{dh}{dt} = 5$, $\frac{dx}{dt} = 15$, gives

$$\frac{dz}{dt} = \frac{15x + 5h}{z}.$$

When $t = 3$, $h = 45 + 3 \times 5 = 60$, $x = 15 \times 3 = 45$, hence $z = \sqrt{45^2 + 60^2} = 75$, and then

$$\frac{dz}{dt} = \frac{15x + 5h}{z} = \frac{15 \times 45 + 5 \times 60}{75} = 13ft/s.$$

Question 7. Use linear approximation to estimate $\frac{1}{4.002}$.

Solution. Let $f(x) = \frac{1}{x}$. Then $f'(x) = -\frac{1}{x^2}$, and $f'(4) = -\frac{1}{16}$. Hence

$$L(x) = \frac{1}{4} - \frac{1}{16}(x-4).$$

So

$$f(4.002) \approx L(0.002) = \frac{1}{4} - \frac{1}{16}(4.002 - 4) = \frac{1}{4} - \frac{1}{8000} = \frac{1999}{8000}.$$

Question 8. Find the absolute maximum and absolute minimum values of $f(x) = \frac{x}{x^2-x+1}$ on $[0,3]$.

Solution. Computing,

$$f'(x) = \frac{1-x^2}{(x^2-x+1)^2} = \frac{(x+1)(1-x)}{(x^2-x+1)^2}.$$

So $f'(1) = 0$ ($f'(-1) = 0$ too, but $-1 \notin [0,3]$). Computing: $f(0) = 0$, $f(1) = 1$, $f(3) = \frac{3}{7}$, thus $f(1) = 1$ is the absolute maximum and $f(0) = 0$ the absolute minimum. To identify what type of point $x = 3$ is, notice that $f'(x) < 0$ for x near and on the left of 3, and we conclude that f has a local minimum at $x = 3$.

Question 9. Find the local and absolute maxima and minima of x^3+x^2-4x-4 on the interval $[-3, 10]$.

Solution. Compute

$$f'(x) = 3x^2 + 2x - 4.$$

Setting equal to zero we find $x = \frac{-1\pm\sqrt{13}}{3}$. Computing $f(-3) = -10$, $f(\frac{-1-\sqrt{13}}{3}) = 0.89$, $f(\frac{-1+\sqrt{13}}{3}) = -6.06$, $f(10) = 1996$. So the absolute minimum occurs at -3 and the absolute maximum at 10. Writing

$$f'(x) = 3(x - \frac{-1-\sqrt{13}}{3})(x - \frac{-1+\sqrt{13}}{3}),$$

and analyzing the sign of f' on the right an left of $\frac{-1\pm\sqrt{13}}{3}$, we conclude that $\frac{-1-\sqrt{13}}{3}$ is a local maximum and $\frac{-1+\sqrt{13}}{3}$ a local minimum.

Question 10. Prove the chain rule.

Solution. This has been done in class, look at your notes.

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