# MATH 155A FALL 13
# PRACTICE MIDTERM 1 — SOLUTIONS.

Question 1. Find the domain of the following functions.

(a) $f(x) = \frac{2x^3-5}{x^2+x-6}$.

(b) $g(x) = \frac{x+1}{1+\frac{1}{x+1}}$.

(c) $f(x) = \sqrt{5-x} + \frac{1}{\sqrt{x-10}}$.

Solution.

(a) We need $x^2 + x - 6 = (x+3)(x-2) \neq 0$. Hence $Dom(f) = \{x \in \mathbb{R} \,|\, x \neq -3, 2\}$.

(b) The denominator of $\frac{1}{x+1}$ needs to be non-zero, thus $x \neq -1$. Also $1 + \frac{1}{x+1} \neq 0$, what gives

$$1 + \frac{1}{x+1} = \frac{x+2}{x+1} \neq 0 \Rightarrow x + 2 \neq 0,$$

or yet $x \neq -2$. Hence $Dom(g) = \{x \in \mathbb{R} \,|\, x \neq -2, -1\}$.

(c) We have $Dom(\sqrt{5-x}) = \{x \in \mathbb{R} \,|\, x \leq 5\}$ and $Dom(\sqrt{x-10}) = \{x \in \mathbb{R} \,|\, x \geq 10\}$. Since the domain of the sum is the intersection of the domains, we have

$$Dom(f) = \{x \in \mathbb{R} \,|\, x \leq 5\} \cap \{x \in \mathbb{R} \,|\, x \geq 10\} = \varnothing,$$

i.e., this $f$ is not well defined.

Question 2. An electricity company charges its customers a base rate of \$10 a month, plust 5 cents per kilowatt-hour (kWh) for the first 1200 kWh and 7 cents per kWh for all usage over 1200 kWh. Express the monthly cost $E$ as a function of the amount $x$ of electricity used.

Solution.

$$E(x) = \begin{cases} 10 + 0.05x, & 0 \leq x \leq 1200 \\ 10 + 0.05 \times 1200 + 0.07(x - 1200), & x > 1200. \end{cases}$$

Question 3. At the surface of the ocean, the water pressure is the same as the air pressure above the water, 15 lb/in$^2$. Below the surface, the water pressure increases by 4.34 lb/in$^2$ for every 10 ft of descent. Express the water pressure as a function of the depth below the ocean surface.

Solution. Let $h$ be the depth below the ocean surface and $P$ the pressure. Then

$$P = 15 + \frac{4.34h}{10},$$

with $P$ measured in lb/in$^2$ and $h$ in ft.

Question 4. Compute the values of the following trigonometric expressions.
(a) $\sin\frac{5\pi}{6}$.
(b) $\tan\frac{19\pi}{4}+\cos(-\frac{\pi}{6})$.
(c) $\sec\frac{4\pi}{3}$.

Solution.
(a) $\frac{1}{2}$. (b) $\tan\frac{19\pi}{4}=\tan(4\pi+\frac{3\pi}{4})=\tan\frac{3\pi}{4}=-1$, $\cos(-\frac{\pi}{6})=\cos(\frac{\pi}{6})=\frac{\sqrt{3}}{2}$. So $\tan\frac{19\pi}{4}+\cos(-\frac{\pi}{6})=\frac{\sqrt{3}}{2}-1$. (c) $\sec\frac{4\pi}{3}=\frac{1}{\cos\frac{4\pi}{3}}=\frac{1}{-\frac{1}{2}}=-2$.

Question 5. Prove the following formulas.
(a) $\sin^2 x-\sin^2 y=\sin(x+y)\sin(x-y)$.
(b) $\cos^2\theta=\frac{1+\cos(2\theta)}{2}$.

Solution.
(a) We have

$$\sin(x\pm y)=\sin x\cos y\pm\sin y\cos x,$$

so

$$\begin{aligned}\sin(x+y)\sin(x-y)&=(\sin x\cos y+\sin y\cos x)(\sin x\cos y-\sin y\cos x)\\&=\sin^2 x\cos^2 y-\sin^2 y\cos^2 x\\&=\sin^2 x(1-\sin^2 y)-\sin^2 y(1-\sin^2 x)\\&=\sin^2 x-\sin^2 x\sin^2 y-\sin^2 y+\sin^2 y\sin^2 x\\&=\sin^2 x-\sin^2 y.\end{aligned}$$

(a) We have

$$\cos(\theta+\xi)=\cos\theta\cos\xi-\sin\theta\sin\xi,$$

so

$$\cos(2\theta)=\cos(\theta+\theta)=\cos^2\theta-\sin^2\theta.$$

Adding the above formula with $1=\cos^2\theta+\sin^2\theta$ yields the result.

Question 6. Find all solutions to the following trigonometric equations.
(a) $2\cos x-1=0$.
(b) $|\tan x|=1$.
(c) $2+\cos 2x=3\cos x$.

Solution.

(a) We have $\cos x = \frac{1}{2}$, which gives $x = \frac{\pi}{3} + 2\pi k$, $x = \frac{5\pi}{3} + 2\pi k$.

(b) We have $\tan x = 1$ or $\tan x = -1$, which gives $x = \frac{\pi}{4} + \pi k$, $x = \frac{3\pi}{4} + \pi k$.

(c) Use $\cos 2x = 2\cos^2 x - 1$ (see question 5) to write

$$2 + 2\cos^2 x - 1 = 3\cos x,$$

or

$$2\cos^2 x - 3\cos x + 1 = 0.$$

This is a quadratic equation for $\cos x$. The quadratic formula gives

$$\cos x = \frac{3 \pm \sqrt{9-8}}{4},$$

so $\cos x = 1$ or $\cos x = \frac{1}{2}$, what gives $x = 0 + 2\pi k$, $x = \frac{\pi}{3} + 2\pi k$, $x = \frac{5\pi}{3} + 2\pi k$.

Question 7. Evaluate the following limits, showing that the limit does not exist when that is the case.

(a) $\displaystyle\lim_{x\to 3^-} \frac{x+2}{x+3}$.

(b) $\displaystyle\lim_{x\to 1} \frac{x^3-1}{\sqrt{x}-1}$.

(c) $\displaystyle\lim_{x\to 2} \sqrt{\frac{2x^3+1}{3x-2}}$.

(d) $\displaystyle\lim_{x\to 0} \sqrt{x^3+x^2}\sin\frac{\pi}{x}$.

(e) $\displaystyle\lim_{x\to \frac{\pi}{2}} |\tan x|$.

(f) $\displaystyle\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{|x|}\right)$.

(g) $\displaystyle\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{|x|}\right)$.

(h) $\displaystyle\lim_{x\to \pi} \sin(x + \sin x)$.

(i) $\lim_{x\to 7}\sqrt{1+\frac{1}{x}}$.

Solution.

(a) The function is defined at 3, so the limit is $\frac{5}{6}$.

(b) Write for $x \neq 1$,

$$\begin{aligned}\frac{x^3-1}{\sqrt{x}-1} &= \frac{x^3-1}{\sqrt{x}-1}\frac{\sqrt{x}+1}{\sqrt{x}+1} = \frac{(\sqrt{x}+1)(x^3-1)}{x-1}\\ &= \frac{(\sqrt{x}+1)(x-1)(x^2+x+1)}{x-1}\\ &= (\sqrt{x}+1)(x^2+x+1).\end{aligned}$$

This expression is defined at $x = 1$ and the limit is therefore $2 \times 3 = 6$.

(c) The function is defined and continuous at $x = 2$, so

$$\lim_{x\to 2}\sqrt{\frac{2x^3+1}{3x-2}} = \sqrt{\lim_{x\to 2}\frac{2x^3+1}{3x-2}} = \sqrt{\frac{17}{4}}$$

(d) Notice that $\lim_{x\to 0}\sqrt{x^3+x^2} = 0$, $\lim_{x\to 0}(-\sqrt{x^3+x^2}) = 0$, and

$$-1 \leq \sin\frac{\pi}{x} \leq 1.$$

Hence

$$-\sqrt{x^3+x^2} \leq \sqrt{x^3+x^2}\sin\frac{\pi}{x} \leq \sqrt{x^3+x^2}.$$

From the squeeze theorem we therefore conclude that

$$\lim_{x\to 0}\sqrt{x^3+x^2}\sin\frac{\pi}{x} = 0.$$

(e) Since $\tan x \to \infty$ when $x \to \frac{\pi}{2}^-$ and $\tan x \to -\infty$ when $x \to \frac{\pi}{2}^+$, we conclude that $|\tan x| \to \infty$ as $x \to \frac{\pi}{2}$.

(f) Because $|x| = x$ for $x > 0$ and the limit is from the right, we can remove the absolute value and then $\frac{1}{x} - \frac{1}{|x|} = \frac{1}{x} - \frac{1}{x} = 0$. Hence the limit is equal to zero.

(g) Because $|x| = -x$ for $x < 0$, and the limit is from the left, if we remove the absolute value: $\frac{1}{x} - \frac{1}{|x|} = \frac{1}{x} + \frac{1}{x} = \frac{2}{x}$ for $x < 0$. Hence

$$\lim_{x\to 0^-}\left(\frac{1}{x} - \frac{1}{|x|}\right) = \lim_{x\to 0^-}\frac{2}{x} = -\infty.$$

We computed the limit from the right above and found zero. Hence the limit does not exist as the limits from the right and left do not agree.

(h) Since $\sin x$ is continuous

$$\lim_{x\to\pi} \sin(x + \sin x) = \sin(\lim_{x\to\pi} + \lim_{x\to\pi} \sin x) = \sin(\pi + \sin\pi) = \sin(\pi + 0) = \sin\pi = 0.$$

(i) Again by continuity

$$\lim_{x\to 7} \sqrt{1 + \frac{1}{x}} = \sqrt{1 + \lim_{x\to 7} \frac{1}{x}} = \sqrt{1 + \frac{1}{7}}.$$

Question 8. Let

$$g(x) = \begin{cases} x & \text{if } x < 1, \\ 3 & \text{if } x = 1, \\ 2 - x^2 & \text{if } 1 < x \leq 2, \\ x - 3 & \text{if } x > 2. \end{cases}$$

Evaluate or explain why the limit does not exist.

(a) $\lim_{x\to 1^-} g(x)$.

(b) $\lim_{x\to 1} g(x)$.

(c) $\lim_{x\to 2^-} g(x)$.

(d) $\lim_{x\to 2+} g(x)$.

(e) $\lim_{x\to 2} g(x)$.

Solution.

(a) 1. (b) 1. (c) $-2$. (d) $-1$. (e) does not exist.

Question 9. For the function $g$ of the previous question, indicate the values of $x$ for which $g$ is not continuous.

Solution. The function is discontinuous at $x = 1$ and $x = 2$.

Question 10. Explain why the following functions are continuous at every point in their domain.

(a) $f(x) = \dfrac{\sin x}{x+1}$.

(b) $f(x) = \dfrac{\tan x}{\sqrt{4-x^2}}$.

(c) $f(x) = \sin(\cos(\sin x))$.

Solution.

(a) $\sin x$ and $x+1$ are continuous. The quotient of continuous functions is continuous whenever the denominator does not vanish.

(b) $\tan x$ is continuous where it is defined, $\sqrt{x}$ is continuous for $x \geq 0$ and $4 - x^2$ is continuous. The composition $\sqrt{4-x^2}$ is therefore continuous where it is defined, since the composition of continuous functions is continuous. The quotient $\frac{\tan x}{\sqrt{4-x^2}}$ is therefore continuous on its domain.

(c) Composition of continuous functions is continuous.

Question 11. Let $f(x) = \frac{x^3-8}{x^2-4}$. Can you define a new function, $g(x)$, which agrees with $f(x)$ on the domain of $f(x)$ and is continuous at $x = 2$? What value should $f(2)$ have if we want to define it as a continuous function at $x = 2$?

Solution. Notice that for $x \neq \pm 2$

$$\frac{x^3-8}{x^2-4} = \frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} = \frac{x^2+2x+4}{x+2}.$$

Define $g$ by the same expression as $f$ for $x \neq \pm 2$, and put $g(2) = \frac{2^2+2\times 2+4}{2+2} = 3$. Also, define $f(2) = 3$.

Question 12. Using the $\varepsilon, \delta$ definition of a limit, show that

(a) $\displaystyle\lim_{x\to 10} (3 - \frac{4}{5}x) = -5$.

(b) $\displaystyle\lim_{x\to -6^+} \sqrt[8]{6+x} = 0$.

Solution.

(a) Write

$$|3 - \frac{4}{5}x - (-5)| = |8 - \frac{4}{5}x| = |8 - \frac{4}{5}(x - 10 + 10)| = \frac{4}{5}|x - 10|.$$

Given $\varepsilon > 0$ we can then choose $\delta = \frac{5}{4}\varepsilon$.

(b) For given $\varepsilon > 0$, we want, for $x > -6$,

$$|\sqrt[8]{6 + x} - 0| = \sqrt[8]{x - (-6)} < \varepsilon.$$

We can therefore choose $\delta = \varepsilon^8$.

Question 13. Using $\varepsilon, \delta$ arguments, prove that the function $f(x) = \frac{1}{x+1}$ is continuous at every point on its domain.

Solution. Fix $a \neq -1$. Given $\varepsilon > 0$, we want

$$\Big|\frac{1}{x+1} - \frac{1}{a+1}\Big| < \varepsilon,$$

or

$$\frac{|x-a|}{|a+1||x+1|} < \varepsilon.$$

If $\delta$ is such that $\delta < \frac{|a+1|}{2}$, then $|x+1| > \frac{|a+1|}{2}$ whenever $|x - a| < \delta$. Thus

$$\frac{|x-a|}{|a+1||x+1|} < \frac{2|x-a|}{|a+1||a+1|} = \frac{2|x-a|}{|a+1|^2}$$

So to get

$$\frac{2|x-a|}{|a+1|^2} < \varepsilon,$$

or equivalently,

$$|x - a| < \frac{|a+1|^2}{2}\varepsilon,$$

we can choose $\delta = \frac{|a+1|^2}{2}\varepsilon$. Therefore, if $\delta = \min\{\frac{|a+1|}{2}, \frac{|a+1|^2}{2}\varepsilon\}$ we conclude that $|x - a| < \delta$ implies

$$\Big|\frac{1}{x+1} - \frac{1}{a+1}\Big| < \varepsilon,$$

what shows that $\frac{1}{x+1}$ is continuous at $a$. Since $a$ is an arbitrary point in the domain, we have the result.

Question 14. Using the definition of derivative, compute $f'(x)$.

(a) $f(x) = x^2$.

(b) $f(x) = \dfrac{1-2x}{3+x}$.

Solution.

(a) Write

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^2 - x^2}{h} \\ &= \frac{x^2 + 2hx + h^2 - x^2}{h} \\ &= \frac{2hx + h^2}{h} = 2x + h.\end{aligned}$$

So

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(2x+h) = 2x.$$

(b) Write

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\frac{1-2(x+h)}{3+(x+h)} - \frac{1-2x}{3+x}}{h} \\ &= \frac{1}{h}\frac{(3+x)(1-2x-2h) - (1-2x)(3+x+h)}{(3+x+h)(3+x)} \\ &= \frac{1}{h}\frac{(3+x)(1-2x) - 2(3+x)h - (1-2x)(3+x) - (1-2x)h}{(3+x+h)(3+x)} \\ &= \frac{1}{h}\frac{-2(3+x)h - (1-2x)h}{(3+x+h)(3+x)} \\ &= \frac{-2(3+x) - (1-2x)}{(3+x+h)(3+x)} = \frac{-7}{(3+x+h)(3+x)}\end{aligned}$$

Hence

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{-7}{(3+x+h)(3+x)} = -\frac{7}{(3+x)^2}.$$

*URL*: `http://www.disconzi.net/Teaching/MAT155A-Fall-13/MAT155A-Fall-13.html`