MATH 155A FALL 13
EXAMPLES SECTION 3.7.

Question. Find the point on the graph of $y = x^2$ that is closest to the point $(18, 0)$.
Solution.

The distance of any point (p, q) to the point $(18, 0)$ is given by

$$D = \sqrt{(p-18)^2 + (q-0)^2}.$$

If (p, q) is on the graph of $y = x^2$, then $p = x$ and $q = x^2$, hence

$$\begin{aligned} D &= \sqrt{(x-18)^2 + (x^2-0)^2} \\ &= \sqrt{x^4 + x^2 - 36x + 324}. \end{aligned}$$

We want to minimize D. Taking the derivative and using the chain rule

$$D' = \frac{4x^3 + 2x - 36}{2\sqrt{x^4 + x^2 - 36x + 324}}.$$

Setting this equal to zero gives

$$2x^3 + x - 18 = 0.$$

By inspection it is seen that $x = 2$ is a solution of the above equation, and since $2x^3 + x - 18$ is an increasing function, we conclude that $x = 2$ is the only solution. To test what type of critical point $x = 2$ is, instead of using the second derivative test, here it is easier to write

$$\begin{aligned} D' &= \frac{4x^3 + 2x - 36}{2\sqrt{x^4 + x^2 - 36x + 324}} \\ &= \frac{2x^3 + x - 18}{\sqrt{x^4 + x^2 - 36x + 324}} \\ &= \frac{(x-2)(2x^2 + 4x + 9)}{\sqrt{x^4 + x^2 - 36x + 324}}. \end{aligned}$$

Since $2x^2 + 4x + 9$ is always positive, we conclude that $D' < 0$ on the left of $x = 2$ and $D' > 0$ on the right of $x = 2$, and thus $x = 2$ is a local minimum of D. Because

$$\lim_{x\to\infty} D = \lim_{x\to\infty} \sqrt{x^4 + x^2 - 36x + 324} = \infty,$$

and

$$\lim_{x\to-\infty} D = \lim_{x\to-\infty} \sqrt{x^4 + x^2 - 36x + 324} = \infty,$$

we conclude that $x = 2$ is the absolute minimum of D.Therefore the distance is minimized when $x = 2$ and $y = 2^2 = 4$.

URL: http://www.disconzi.net/Teaching/MAT155A-Fall-13/MAT155A-Fall-13.html