MATH 155A FALL 13
EXAMPLES SECTION 3.4.

Question. Find the horizontal and vertical asymptotes of

$$f(x) = \frac{\sqrt[3]{x^5 + x^2 - 5}}{x(\sqrt[3]{x^2} - 2)}.$$

Solution.

Write

$$\begin{aligned}
f(x) &= \frac{\sqrt[3]{x^5 + x^2 - 5}}{x(\sqrt[3]{x^2} - 2)} = \frac{\sqrt[3]{x^5(1 + \frac{1}{x^3} - \frac{5}{x^5})}}{x(x^{\frac{2}{3}} - 2)} \\
&= \frac{\sqrt[3]{x^5}\sqrt[3]{1 + \frac{1}{x^3} - \frac{5}{x^5}}}{x^{\frac{5}{3}} - 2x} = \frac{x^{\frac{5}{3}}}{x^{\frac{5}{3}}} \frac{\sqrt[3]{1 + \frac{1}{x^3} - \frac{5}{x^5}}}{1 - \frac{2x}{x^{\frac{5}{3}}}} \\
&= \frac{\sqrt[3]{1 + \frac{1}{x^3} - \frac{5}{x^5}}}{1 - \frac{2}{x^{\frac{2}{3}}}}
\end{aligned}$$

Since

$$\lim_{x\to\infty} \frac{1}{x^3} = 0, \ \lim_{x\to\infty} \frac{5}{x^5} = 0, \ \lim_{x\to\infty} \frac{2}{x^{\frac{2}{3}}} = 0, \ \text{ and}$$
$$\lim_{x\to-\infty} \frac{1}{x^3} = 0, \ \lim_{x\to-\infty} \frac{5}{x^5} = 0, \ \lim_{x\to-\infty} \frac{2}{x^{\frac{2}{3}}} = 0,$$

we obtain

$$\lim_{x\to\infty} \frac{\sqrt[3]{x^5 + x^2 - 5}}{x(\sqrt[3]{x^2} - 2)} = \lim_{x\to\infty} \frac{\sqrt[3]{1 + \frac{1}{x^3} - \frac{5}{x^5}}}{1 - \frac{2}{x^{\frac{2}{3}}}} = 1,$$

and

$$\lim_{x\to-\infty} \frac{\sqrt[3]{x^5 + x^2 - 5}}{x(\sqrt[3]{x^2} - 2)} = \lim_{x\to-\infty} \frac{\sqrt[3]{1 + \frac{1}{x^3} - \frac{5}{x^5}}}{1 - \frac{2}{x^{\frac{2}{3}}}} = 1.$$

Thus $y = 1$ is a horizontal asymptote.

For the vertical asymptotes, notice that

$$\lim_{x\to 0^-} \frac{\sqrt[3]{x^5 + x^2 - 5}}{x(\sqrt[3]{x^2} - 2)} = -\infty,$$

and

$$\lim_{x\to 0^+} \frac{\sqrt[3]{x^5+x^2-5}}{x(\sqrt[3]{x^2}-2)} = \infty.$$

Hence $x=0$ is a vertical asymptote.

We also have

$$\lim_{x\to 2^{\frac{3}{2}}\,-} \frac{\sqrt[3]{x^5+x^2-5}}{x(\sqrt[3]{x^2}-2)} = -\infty,$$

and

$$\lim_{x\to 2^{\frac{3}{2}}\,+} \frac{\sqrt[3]{x^5+x^2-5}}{x(\sqrt[3]{x^2}-2)} = \infty,$$

showing that $x = 2^{\frac{3}{2}}$ is a vertical asymptote.

Finally, compute

$$\lim_{x\to -2^{\frac{3}{2}}\,-} \frac{\sqrt[3]{x^5+x^2-5}}{x(\sqrt[3]{x^2}-2)} = \infty,$$

and

$$\lim_{x\to -2^{\frac{3}{2}}\,+} \frac{\sqrt[3]{x^5+x^2-5}}{x(\sqrt[3]{x^2}-2)} = -\infty,$$

showing that $x = -2^{\frac{3}{2}}$ is also a vertical asymptote.

URL: `http://www.disconzi.net/Teaching/MAT155A-Fall-13/MAT155A-Fall-13.html`