MATH 155A FALL 13
EXAMPLES SECTIONS 1.7 AND 1.8.

Question 1. What value should A have in order to make the function

$$f(x) = \begin{cases} x^2 - 7, & x \leq 5, \\ A\cos(\frac{\pi}{15}x) - 3, & x > 5, \end{cases}$$

continuous?

Question 2. Show that the equation

$$\cos^2 x = x^3,$$

has at least one solution.

Question 3. Using the ε, δ definition of limit, find

$$\lim_{x \to -2} (3x + 4).$$

Solution 1. Since $x^2 - 7$ and $A\cos(\frac{\pi}{15}x) - 3$ are continuous for any value of A, we only need to check continuity at $x = 5$.

From the definition of f we have

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (x^2 - 7) = 5^2 - 7 = 18,$$

and

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} (A\cos(\frac{\pi}{15}x) - 3) = A\cos(\frac{\pi}{15} \times 5) - 3 = A\cos(\frac{\pi}{3}) - 3 = \frac{A}{2} - 3.$$

Notice that it is also true that $f(5) = 18$. For f to be continuous at 5 we need

$$\lim_{x \to 5^-} f(x) = f(5) = \lim_{x \to 5^+} f(x).$$

Hence,

$$18 = \frac{A}{2} - 3 \Rightarrow A = 42.$$

Solution 2. Define $f(x) = \cos^2 x - x^3$, and notice that a point c is a solution of the equation if and only if $f(c) = 0$. We have

$$f(0) = \cos^2 0 - 0^3 = 1,$$

and

$$f(1) = \cos^2 1 - 1^3 < 0.$$

To see why the last inequality is true, notice first that 1 lies between zero and $\frac{\pi}{2}$ (recall that $\pi \approx 3.14$, thus $\frac{\pi}{2} \approx \frac{3.14}{2} \approx 1.5 > 1$). Since $\cos x$ is decreasing between zero and $\frac{\pi}{2}$ (recall the graph of $\cos x$) and $\cos 0 = 1$, we conclude that $\cos 1 < 1$. But from this it follows that $\cos 1 - 1 < 0$.

Next, observe that $f(x)$ is continuous. Therefore, since $f(0) = 1$ and $f(1) < 0$, by the intermediate value theorem we conclude that there exists a point $c \in (0,1)$ such that $f(c) = 0$, as desired.

Solution 3. To use ε, δ arguments, we should first have an idea of what the limit is. But we readily see that

$$\lim_{x \to -2} (3x + 4) = -2.$$

Therefore, the statement to be shown is the following: given $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $|x - (-2)| < \delta$, then $|3x + 4 - (-2)| < \varepsilon$.

Write

$$\begin{aligned} |3x + 4 - (-2)| &= |3x + 6| = |3\big(x - (-2) + (-2)\big) + 6| \\ &= |3(x+2) - 6 + 6| = 3|x + 2| < \varepsilon. \end{aligned}$$

Therefore, if we choose $\delta = \frac{\varepsilon}{3}$, we have that

$$|x - (-2)| < \delta \Rightarrow |3x + 4 - (-2)| < \varepsilon.$$

URL: `http://www.disconzi.net/Teaching/MAT155A-Fall-13/MAT155A-Fall-13.html`