THE CHAIN RULE

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1. Why do we need it?

Here I want to give further explanation to the chain rule. The basic idea is the following. We know how to differentiate simple function like e^x:

$$(e^x)' = e^x \tag{1.1}$$

or x^3:

$$(x^3)' = 3x^2$$

where $'$ means derivative, as usual (if the two formulas above don't make sense to you, go back to your class notes and review the basic of derivatives before reading these notes).

The point with the previous differentiation formulas (and all formulas in general that you memorize for the exams) is that they don't tell you how to compute derivatives of *similar* functions. For instance, if you are asked to compute the derivative of e^{3x}, you might be tempted to say:

$$(e^{3x})' = e^{3x} \tag{1.2}$$

But this is **wrong**: formula (1.1) tells you how to compute the derivative of e^x; it doesn't say anything about the derivative of e^{3x}. In other words, you need to be very strict when using these type of formulas: if you have something different than e^x, you are not allowed to make up your own rules by analogy.

To compute the derivative of functions like e^{3x} we need to use the chain rule. More formally, the chain rule is what we use to compute the derivative of *composite* functions; e.g., e^{3x} is a composition of e^x with $3x$. If you don't remember what the composition of two functions is, try to understand that before learning the chain rule.

2. How to use

The chain rule allows us to compute derivatives of compositions. In a nutshell, a composite function has an "outside" function f and an "inside" function g. We write this in the form:

$$h(x) = f(g(x))$$

So, for the function e^{3x}, we write $h(x) = e^{3x}$, the outside function is the exponential e^x and the inside is $3x$, i.e., $f(x) = e^x$ and $g(x) = 3x$. Notice that we know how to compute the derivatives of f and g separately, namely:

$$f'(x) = (e^x)' = e^x$$

and

$$g'(x) = (3x)' = 3$$

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This will always be the case when applying the chain rule: we start with a function like e^{3x} for which we cannot apply directly one of the derivative formulas, and then write it as a composition of two functions f and g *each of which we know how to compute the derivative separately.*

The chain rule then says:

$$h'(x) = f'(g(x))g'(x) \tag{2.1}$$

In words: "the derivative of the outside times the derivative of the inside". It's **important** to notice that after computing the derivative of f we need to plug $g(x)$ in f'. In the current example: $f'(x) = e^x$, but instead of x we need to plug in $g(x) = 3x$, so $f'(g(x)) = e^{3x}$. The derivative of g is $g'(x) = 3$, so the final answer, using (2.1), is:

$$(e^{3x})' = e^{3x} \cdot 3 = 3e^{3x}$$

Notice that this is different from the wrong answer (1.2).

3. Step by step

Let's do another example, indicating a a step-by-step procedure

Compute the derivative of $\ln(\sqrt{x})$.

Step 1. Identify the outside and the inside functions. Recall that you need to know how to compute separately the derivatives of the inside and outside. In this example the outside function is:

$$f(x) = \ln x$$

and the inside is:

$$g(x) = \sqrt{x}$$

Step 2. Compute f' and g':

$$f'(x) = (\ln x)' = \frac{1}{x}$$

and:

$$g'(x) = (\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

(you are supposed to know that $(\ln x)' = \frac{1}{x}$ and $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$. If you don't remember this, review the derivatives of basic functions).

Step 3. Change x in $f'(x)$ to $g(x)$:

$$f'(x) = \frac{1}{x} \mapsto f'(g(x)) = \frac{1}{\sqrt{x}}$$

Step 4. Use formula (2.1) to get the result:

$$(\ln(\sqrt{x}))' = \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x}$$