VANDERBILT UNIVERISTY
MATH 8110 — THEORY OF PARTIAL DIFFERENTIAL EQUATIONS
HW 1

1. Problems

The notation and terminology below is the same used in class.

Problem 1. Verify whether the given function is a solution of the given PDE:

(a) $u(x, y) = y \cos x + \sin y \sin x$, $u_{xx} + u = 0$.

(b) $u(x, y) = \cos x \sin y$, $(u_{xx})^2 + (u_{yy})^2 = 0$.

Problem 2. For each PDE seen as example in the first class (Laplace's equation, Helmholtz's equation, linear transport equation, heat equation, Schödinger's equation, wave equation, eikonal equation, minimal surface equation, Burgers' equation, Maxwell's equation, Euler and Navier-Stokes equations, Einstein's equations), state whether it is a scalar PDE (i.e., single PDE) or a system of PDEs, its order, and whether it is a linear or non-linear PDE.

Problem 3. Write each PDE below in the form $P(D^k u, \ldots, Du, u, x) = 0$, i.e., identify the function P. State if the PDE is homogeneous or non-homogeneous, linear or non-linear.

(a) $u_{tt} - u_{xx} = f$.

(b) $u_y + uu_x = 0$.

(c) $a^{ijk}\partial^3_{ijk} v + v = 0$,
where i, j, k range from 1 to 3.

(d) $u_{xx} + x^2 y^2 u_{yy} = (x + y)^2$.

(e) $u_{xy} + \cos(u) = \sin(xy)$.

Problem 4. Consider a linear homogeneous PDE. Explain why any linear combination of solutions is also a solution. (Again, use your knowledge of ODE to define linearity here.)

Problem 5. Consider Maxwell's equations:

$$\begin{aligned}
\operatorname{div} E &= \frac{\varrho}{\varepsilon_0}, \\
\operatorname{div} B &= 0, \\
\frac{\partial B}{\partial t} + \operatorname{curl} E &= 0, \\
\frac{\partial E}{\partial t} - \frac{1}{\mu_0 \varepsilon_0} \operatorname{curl} B &= -\frac{1}{\varepsilon_0} J.
\end{aligned}$$

Assume that ϱ and J vanish. Show that Maxwell's equations then imply that E and B satisfy the wave equation:

$$\frac{\partial^2 E}{\partial t^2} - \frac{1}{\varepsilon_0 \mu_0} \Delta E = 0,$$

and

$$\frac{\partial^2 B}{\partial t^2} - \frac{1}{\varepsilon_0 \mu_0} \Delta B = 0.$$

Interpret your result. Can you guess what the constant $\frac{1}{\varepsilon_0 \mu_0}$ must equal to?

Problem 6. Consider Euler's equations:

$$\partial_t \varrho + u^i \partial_i \varrho + \varrho \partial_i u^i = 0,$$
$$\varrho(\partial_t u^j + u^i \partial_i u^j) + \nabla^j p = 0,$$

where we recall that $p = p(\varrho)$. A fluid is called *incompressible* if $\varrho = constant$, in which case we can set $\varrho = 1$. In this case, the equations describing the fluid motion are

$$\partial_t u^j + u^i \partial_i u^j + \nabla^j p = 0,$$
$$\partial_i u^i = 0,$$

which are called the *incompressible Euler equations.* For an incompressible fluid, however, the pressure is no longer given by $p = p(\varrho)$, since the pressure would then be constant, but experiments show that the pressure can vary even if the density remains (approximately) constant. Show that in the case of the incompressible Euler equations, the pressure is given as a solution to

$$\Delta p = -\partial_j u^i \partial_i u^j.$$

Problem 7. Consider the incompressible Euler equations (see previous question):

$$\partial_t u^j + u^i \partial_i u^j + \nabla^j p = 0,$$
$$\partial_i u^i = 0.$$

The *vorticity* ω of the fluid is defined as

$$\omega := \operatorname{curl} u.$$

The vorticity is an important physical quantity; it measures, as the name suggests, "eddies" in the fluid. It is, therefore, important to know how it changes in time and space (i.e., what the dynamics of the vorticity is). Show that ω satisfies the following PDE:

$$\partial_t \omega + \nabla_u \omega - \nabla_\omega u = 0.$$

Above, the operators ∇_u and ∇_ω are defined as follows. For any vector field X, ∇_X is a short hand notation for $X \cdot \nabla$, i.e.,

$$\nabla_X := X \cdot \nabla,$$

where we recall that $X \cdot \nabla$ has been defined in class as

$$X \cdot \nabla = X^i \partial_i.$$

2. Solutions

Solution 1. (a) Compute $u_{xx}(x, y) = -y \cos x - \sin x \sin y = -u(x, y)$, thus u is a solution.

(b) Compute $u_{xx}(x, y) = -\cos x \sin y$, $u_{yy}(x, y) = -\cos x \sin y$, thus

$$u_{xx}(x, y))^2 + (u_{yy}(x, y))^2 = 2 \cos^2 x \sin^2 y \neq 0,$$

hence u is not a solution.

Solution 2. Laplace's equation: scalar, second order, linear. Helmholtz's equation: scalar, second order, linear. Linear transport equation: scalar, first order, linear. Heat equation: scalar, second order (first-order in time), linear. Schödinger's equation: complex scalar, second order (first-order in time), linear. Wave equation: scalar, second order, linear. Eikonal equation: scalar, first order, fully nonlinear. Minimal surface equation: scalar, second order, quasi-linear. Burgers' equation: scalar, first order, quasi-linear. Maxwell's equation: system, first order, linear. Euler's equations: system, first order, quasi-linear. Navier-Stokes' equations: system, second order (first-order in time), quasi-linear. Einstein's equations: it's complicated, but when written in a specific "gauge," it's a system, second order, quasi-linear.

Solution 3. In order to find F, it is useful to identify whether the PDE is linear, homogeneous, the unknown function, etc.

(a) Unknown: u. Independent variables: x, t. Order: second. We have

$$P(p_1, \ldots, p_9) = p_9 - p_6 - f(p_1, p_2).$$

The equation is linear and non-homogeneous.

(b) Unknown: u. Independent variables: x, y. Order: first. We have

$$P(p_1, \ldots, p_5) = p_5 + p_3 p_4.$$

The equation is non-linear (because of the term uu_x).

(c) It is instructive to consider a slightly more general case, with i, j, k ranging from 1 to n. Unknown: v. Independent variables: $x^1, \ldots, x^n$. Order: third. We have

$$P(x_1, \ldots, x_n, p, p_1, \ldots, p_n, p_{11}, \ldots, p_{nn}, \ldots, p_{111}, \ldots, p_{nnn}) = a^{ijk} p_{ijk} + p.$$

The equation is linear and homogeneous.

(d) Unknown: u. Independent variables: x, y. Order: second. We have

$$P(p_1, \ldots, p_9) = p_6 + p_1^2 p_2^2 p_9 - (p_1 + p_2)^2.$$

The equation is linear and non-homogeneous.

(e) Unknown: u. Independent variables: x, y. Order: second. We have

$$P(p_1, \ldots, p_9) = p_7 + \cos p_3 - \sin(p_1 p_2).$$

The equation is non-linear (because of $\cos u$).

Solution 4. Sums and multiplication by constants are preserved by linearity.

Solution 5. Under the assumptions, the equations become

$$\text{div} E = 0, \tag{2.1}$$

$$\text{div} B = 0, \tag{2.2}$$

$$\frac{\partial B}{\partial t} + \text{curl} E = 0, \tag{2.3}$$

$$\frac{\partial E}{\partial t} - \frac{1}{\mu_0 \varepsilon_0} \text{curl} B = 0. \tag{2.4}$$

Take the curl of (2.3) and note that $\text{curl} \frac{\partial}{\partial t} = \frac{\partial}{\partial t} \text{curl}$ to get

$$\frac{\partial}{\partial t} \text{curl} B + \text{curl} \text{curl} E = 0.$$

But $\text{curl} B = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$ by (2.4), thus

$$\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} + \text{curl} \text{curl} E = 0.$$

Recalling the following identity from multivariable calculus

$$\mathrm{curl}\mathrm{curl} f = \nabla(\mathrm{div} f) - \Delta f,$$

and using (2.1), we obtain the wave equation for E. The wave equation for B is similarly obtained.

The interpretation is that the electric and magnetic fields propagate in vacuum as waves. From the discussion about the wave equation in class, we conclude that $\frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ is the speed of propagation of the electromagnetic waves, which, from physics, we know to be equal to the speed of light (in vacuum).

Solution 6. Taking the divergence of the momentum equation and using that $\partial_i u^i = 0$, we find

$$\begin{aligned} 0 &= \partial_j(\partial_t u^j + u^i \partial_i u^j + \nabla^j p) \\ &= \partial_t \partial_j u^j + \partial_j u^i \partial_i u^j + u^i \partial_i \partial_j u^j + \partial_i \partial^i p \\ &= \partial_j u^i \partial_i u^j + \partial_i \partial^i p, \end{aligned}$$

where we denoted $\partial^i := \delta^{ij}\partial_j$, with δ being the Kronecker-delta symbol defined as $\delta^{ij} = \delta_{ij} = \delta^i_j = 1$ if $i = j$ and 0 otherwise. Noting that $\partial^i \partial_i = \Delta$, we have the result.

Remark. Note that while Euler's equations in principle require functions that are only once differentiable, the above calculation assumed that the functions are in fact twice continuously differentiable.

Solution 7. Denoting by $|\cdot|$ the norm in $\mathbb{R}^3$, observe the following identity:

$$\frac{1}{2}\nabla^i |u|^2 = \frac{1}{2}\nabla^i (u^\ell u_\ell) = u^\ell \partial^i u_\ell = u^\ell \partial_\ell u^i + (u^\ell \partial^i u_\ell - u^\ell \partial_\ell u^i),$$

where ∂^i is as in the last question. Next, compute

$$\begin{aligned} (u \times \omega)^i &= \epsilon^{ijk} u_i \omega_k = \epsilon^{ijk} u_j \epsilon_k{}^{\ell n} \partial_\ell u_n \\ &= (\delta^{i\ell}\delta^{jn} - \delta^{j\ell}\delta^{in}) u_j \partial_\ell u_n \\ &= u^n \partial^i u_n - u^\ell \partial_\ell u^i, \end{aligned}$$

where we used the identity

$$\epsilon^{ijk}\epsilon_{k\ell n} = \epsilon^{kij}\epsilon_{k\ell n} = \delta^i_\ell \delta^j_n - \delta^j_\ell \delta^i_n,$$

which can be verified directly. From the foregoing we conclude that

$$\nabla_u u = \frac{1}{2}\nabla |u|^2 - u \times \omega,$$

which implies

$$\mathrm{curl} \nabla_u u = -\mathrm{curl}(u \times \omega).$$

Let us compute the RHS:

$$\begin{aligned} (\mathrm{curl}(u \times \omega))^i &= \epsilon^{ijk}\partial_j \omega_k = \epsilon^{ijk}\partial_j(\epsilon_k{}^{\ell n}\partial_\ell u_n) \\ &= \epsilon^{ijk}\epsilon_k{}^{\ell n}\partial_j u_\ell \omega_n + \epsilon^{ijk}\epsilon_k{}^{\ell n} u_\ell \partial_j \omega_n \\ &= (\delta^{i\ell}\delta^{jn} - \delta^{j\ell}\delta^{in})\partial_j u_\ell \omega_n + (\delta^{i\ell}\delta^{jn} - \delta^{j\ell}\delta^{in}) u_\ell \partial_j \omega_n \\ &= \partial^n u^i \omega_n - \underbrace{\partial_\ell u^\ell}_{=0} \omega^i + u^i \underbrace{\partial_n \omega^n}_{=0} - u^j \partial_j \omega^i \\ &= (\nabla_\omega u)^i - (\nabla_u \omega)^i, \end{aligned}$$

which implies the result.

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HW 2

1. Problems

Unless stated otherwise, the notation below is as in class.

Problem 1. Show that Laplace's equation is rotationally invariant, i.e., if u solves $\Delta u = 0$ and we define

$$\tilde{u}(x) = u(Mx),$$

where M is an orthogonal matrix, then $\Delta \tilde{u} = 0$.

Problem 2. Prove the following fact that we used in the construction of solutions to Poisson's equation: let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous, then

$$\lim_{r \to 0^+} \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} f \, dS = f(x).$$

Hint: Consider the difference $f(x) - \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} f \, dS$ and use $\frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} dS = 1$.
Remark: The result is valid under weaker assumptions; in fact, it holds for a.e. x_0 if f is assumed to be locally integrable (this is sometimes known as the Lebesgue differentiation theorem).

Problem 3. In class, we constructed solutions to Poisson's equation in $\mathbb{R}^n$ for $n \geq 3$. Carry out the construction in the case $n = 2$. You do *not* have to do all the steps. Rather, follow what was done in class and point out what changes in $n = 2$. This boils down to slightly modifying some of the estimates for the fundamental solution.

Problem 4. Let u be a non-trivial harmonic function in $\mathbb{R}^n$. Can u have compact support?
Hint: mean value theorem.

Problem 5. Prove the converse of the mean value theorem. I.e., let $u \in C^2(\Omega)$ be such that

$$u(x) = \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u \, dS$$

for each $B_r(x) \subset\subset \Omega$. Show that $\Delta u = 0$ in Ω.
Hint: Assume that $\Delta u(x) \neq 0$ for some $x \in \Omega$. Use the functions $f(r)$, $f'(r)$ used in the proof of the mean value to derive a contradiction.

2. Solutions

Solution 1. Write $y = Mx$. The chain rule gives

$$\begin{aligned}\frac{\partial}{\partial x^i} &= \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j} \\ &= M_{ji}\frac{\partial}{\partial y^j},\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2}{\partial(x^i)^2} &= \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i} \\
&= \left(M_{ji}\frac{\partial}{\partial y^j}\right)\left(M_{\ell i}\frac{\partial}{\partial y^\ell}\right) \\
&= M_{ji}M_{\ell i}\frac{\partial^2}{\partial y^j\partial y^\ell},
\end{aligned}$$

where there is no sum over i above. Summing over i:

$$\begin{aligned}
\Delta_x &= \sum_i \frac{\partial^2}{\partial(x^i)^2} \\
&= \sum_i M_{ji}M_{\ell i}\frac{\partial^2}{\partial y^j\partial y^\ell} \\
&= \delta^j_\ell \frac{\partial^2}{\partial y^j\partial y^\ell} \\
&= \sum_j \frac{\partial^2}{\partial(y^j)^2} \\
&= \Delta_y,
\end{aligned}$$

where we used that $MM^T = I$, i.e.,

$$\sum_i M_{ji}M_{\ell i} = \delta_{j\ell}.$$

Solution 2. We have to prove that given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < r < \delta$ then

$$\left|\frac{1}{\mathrm{vol}(\partial B_r(x))}\int_{\partial B_r(x)} f\,dS - f(x)\right| < \varepsilon.$$

Write

$$\begin{aligned}
\frac{1}{\mathrm{vol}(\partial B_r(x))}\int_{\partial B_r(x)} f(y)\,dS(y) - f(x) &= \frac{1}{\mathrm{vol}(\partial B_r(x))}\int_{\partial B_r(x)} f(y)\,dS - \frac{f(x)}{\mathrm{vol}(\partial B_r(x))}\int_{\partial B_r(x)} dS(y) \\
&= \frac{1}{\mathrm{vol}(\partial B_r(x))}\int_{\partial B_r(x)} (f(y) - f(x))\,dS(y).
\end{aligned}$$

Thus

$$\left|\frac{1}{\mathrm{vol}(\partial B_r(x))}\int_{\partial B_r(x)} f(y)\,dS(y) - f(x)\right| \le \frac{1}{\mathrm{vol}(\partial B_r(x))}\int_{\partial B_r(x)} |f(y) - f(x)|\,dS(y).$$

Fix $\varepsilon > 0$. Since f is continuous, there exists a $\delta > 0$ such that if $|x-y| < \delta$ then $|f(x)-f(y)| < \varepsilon$. If $r < \delta$, then $|x-y| < \delta$ for all $y \in \partial B_r(x)$, thus

$$\left|\frac{1}{\mathrm{vol}(\partial B_r(x))}\int_{\partial B_r(x)} f(y)\,dS(y) - f(x)\right| < \frac{1}{\mathrm{vol}(\partial B_r(x))}\int_{\partial B_r(x)} \varepsilon\,dS = \varepsilon.$$

Solution 3. We use the following estimates in the $n = 2$ case:

$$\int_{B_\varepsilon(0)} |\Gamma(y)|\,dy \le C\varepsilon^2|\ln\varepsilon| \to 0 \text{ as } \varepsilon \to 0^+,$$

and

$$\int_{\partial B_\varepsilon(0)} |\Gamma(y)|\, dS(y) \le C\varepsilon |\ln \varepsilon| \to 0 \text{ as } \varepsilon \to 0^+,$$

and the rest of the proof is essentially the same.

Solution 4. No. Let u be harmonic and with compact support and fix an arbitrary $x \in \mathbb{R}^n$. By the compact support property, there exists a $r > 0$ such that $u(y) = 0$ for all $y \in \partial B_r(x)$. By the mean value formula

$$u(x) = \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u(y)\, dS(y) = 0,$$

so that $u = 0$ since x is arbitrary.

Solution 5. If u is not harmonic, there exists a $x \in \Omega$ such that $\Delta u(x) \neq 0$. By assumption, the function

$$f(r) = \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u\, dS$$

is constant equal to $u(x)$ on the interval $(0, R)$, where $R > 0$ is a fixed number such that $B_R(x) \subset \Omega$. Thus $f'(r) = 0$ for all $r \in (0, R)$. On the other hand, by continuity, Δu has a definite sign, say positive, on a ball $B_{r_0}(x)$ for some $r_0 > 0$, which without loss of generality we can take such that $r_0 < R$. Arguing as in the proof of the mean value theorem, we find

$$f'(r_0) = \frac{1}{n\omega_n r_0^{n-1}} \int_{B_{r_0}(x)} \Delta u(y)\, dy > 0,$$

contradicting $f'(r_0) = 0$.

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HW 3

Unless stated otherwise, the notation below is as in class.

1. Problems

Problem 1. Prove that harmonic functions are analytic.

Problem 2. Prove Liouville's theorem for harmonic functions in $\mathbb{R}^n$.

Problem 3. Prove Harnack's inequality for (non-negative) harmonic functions.

The remaining questions are about the heat equation in n-dimensions, i.e.,

$$u_t - \Delta u = 0 \text{ in } (0,\infty)\times\mathbb{R}^n. \tag{1.1}$$

Problem 4. Look for a solution to (1.1) in the form

$$u(t,x) = t^{-\alpha} v(t^{-\beta}x), \tag{1.2}$$

where α and β will be chosen and v will be determined. More precisely, proceed as follows:

(a) Show that plugging (1.2) into (1.1) produces

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y\cdot\nabla v(y) + t^{-(\alpha+2\beta)}\Delta v(y) = 0, \tag{1.3}$$

where $y := t^{-\beta}x$.

(b) Set $\beta = \frac{1}{2}$ in (1.3) to obtain

$$\Delta v(y) + \frac{1}{2} y\cdot\nabla v(y) + \alpha v(y) = 0. \tag{1.4}$$

(c) Assume that v is radially symmetric, i.e.,

$$v(y) = w(r), \tag{1.5}$$

where w is to be determined. Show that in this case (1.4) becomes

$$w'' + \frac{n-1}{r}w' + \frac{1}{2}rw' + \alpha w = 0. \tag{1.6}$$

(d) Set $\alpha = \frac{n}{2}$ in (1.6) to find

$$(r^{n-1}w')' + \frac{1}{2}(r^n w)' = 0. \tag{1.7}$$

(e) From (1.7), conclude that

$$r^{n-1}w' + \frac{1}{2}r^n w = A, \tag{1.8}$$

where A is a constant.

(f) Set $A = 0$ in (1.8) and conclude that

$$w(r) = Be^{-\frac{1}{4}r^2}, \tag{1.9}$$

where B is a constant.

(g) Combine (1.2), (1.5), (1.9), and take into account the choices of α and β, to conclude that

$$u(t,x) = \frac{B}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}},\ t > 0, \tag{1.10}$$

is a solution to (1.1).

Problem 5. Recall that

$$\Gamma(t,x) := \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, & t > 0, x \in \mathbb{R}^n, \\ 0, & t < 0, x \in \mathbb{R}^n, \end{cases}$$

is called the *fundamental solution of the heat equation.* Note that for $t > 0$, $\Gamma(t,x)$ is simply (1.10) with a specific choice of the constant B. In particular, $\Gamma(t,x)$ is a solution of (1.1).

This choice of B is to guarantee Γ to integrate to 1, i.e., using the fact that

$$\int_{\mathbb{R}^n} e^{-|x|^2}\,dx = \pi^{\frac{n}{2}}, \tag{1.11}$$

show that for each $t > 0$

$$\int_{\mathbb{R}^n} \Gamma(t,x)\,dx = 1.$$

(You do *not* have to show (1.11).)

Problem 6. Consider the initial-value problem for the heat equation:

$$u_t - \Delta u = 0, \qquad \text{in } (0,\infty) \times \mathbb{R}^n, \tag{1.12a}$$

$$u(0,x) = g(x),\ \ x \in \mathbb{R}^n. \tag{1.12b}$$

In (1.12), assume that $g \in C^0(\mathbb{R}^n)$ and that there exists a constant $C > 0$ such that $|g(x)| \le C$ for all $x \in \mathbb{R}^n$.

Recall that we showed existence of a solution by defining

$$u(t,x) := \int_{\mathbb{R}^n} \Gamma(t, x-y) g(y)\,dy,\ t > 0, x \in \mathbb{R}^n. \tag{1.13}$$

Show that (1.13) is well-defined.

Problem 7. Provide the details of the proof given in class that $u \in C^\infty((0,\infty) \times \mathbb{R}^n)$, where u is defined by (1.13).

Hint: Use the following fact, that you do *not* need to prove. Let α be a multiindex and $t > 0$. If

$$\int_{\mathbb{R}^n} D_x^\alpha \Gamma(t, x-y) g(y)\,dy$$

is well-defined, then

$$D^\alpha u(t,x) = \int_{\mathbb{R}^n} D_x^\alpha \Gamma(t, x-y) g(y)\,dy,$$

where we write D_x^α on the RHS to emphasize that the differentiation is with respect to the x variable.

Problem 8. Look up the mean value formula and the maximum principle for solutions to the heat equation.

2. Solutions

Solution 1. See section 2.2.3 of Evan's book.

Solution 2. See section 2.2.3 of Evan's book.

Solution 3. See section 2.2.3 of Evan's book.

Solution 4. These are a sequence of straightforward calculations that are done in the class notes.

Solution 5. Set $z = x/\sqrt{4t}$ and change variables to find

$$\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}}\, dx = \int_{\mathbb{R}^n} e^{-|z|^2} (\sqrt{4t})^n \, dz = \pi^{\frac{n}{2}} (4t)^{\frac{n}{2}}.$$

Solution 6. We have

$$|u(t,x)| \leq \frac{C}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}\, dy.$$

Making the change of variables $z = (y-x)/\sqrt{4t}$ we find

$$\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}\, dy = (4t)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-|z|^2}\, dz < \infty.$$

Solution 7. Let $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)$ be an arbitrary multiindex. Then

$$D_x^\alpha \Gamma(t, x-y) = \frac{p(t,x,y)}{t^M} e^{-\frac{|x-y|^2}{4t}}, \tag{2.1}$$

where M is a non-negative constant and p is a polynomial on its arguments (If (2.1) is not clear, take a few derivatives of $\Gamma(t, x-y)$ and see the pattern that emerges.) Then, using the assumption on g,

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} D_x^\alpha \Gamma(t,x-y) g(y)\, dy \right| &\leq C \int_{\mathbb{R}^n} |D_x^\alpha \Gamma(t,x-y)|\, dy \\
&\leq C \int_{\mathbb{R}^n} \frac{|p(t,x,y)|}{t^M} e^{-\frac{|x-y|^2}{4t}}\, dy \\
&= \int_{\mathbb{R}^n} \frac{|q(t,x,z)|}{t^N} e^{-|z|^2}\, dz,
\end{aligned}$$

where in the last step we changed variables $z = (y-x)/\sqrt{4t}$, N is a non-negative constant, and q is polynomial on its arguments. We claim that there exists a constant $C > 0$, possibly depending on t, such that

$$\frac{|q(t,x,z)|}{t^N} e^{-|z|^2} \leq C e^{-\frac{1}{2}|z|^2}. \tag{2.2}$$

For, (2.2) is equivalent to

$$\frac{|q(t,x,z)|}{t^N} e^{-\frac{1}{2}|z|^2} \leq C. \tag{2.3}$$

For each fixed x and $t > 0$, the function $\frac{|q(t,x,z)|}{t^N} e^{-\frac{1}{2}|z|^2}$ is a continuous function of z, and because the exponential decays faster than any polynomial, we conclude that $\frac{|q(t,x,z)|}{t^N} e^{-\frac{1}{2}|z|^2}$ is bounded in $\mathbb{R}^n$ as a function of z for each fixed x and $t > 0$, which is (2.3). Since the integral of $e^{-\frac{1}{2}|z|^2}$ is finite, we have shown the result in view of the hint and the fact that α, x, and $t > 0$ are arbitrary.

Solution 8. See sections 2.3.2 and 2.3.3 of Evan's book.

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HW 4

Unless stated otherwise, the notation below is as in class. You can assume that all functions are C^∞ unless stated otherwise.

1. Problems

Problem 1. Prove the differentiation of moving regions formula stated in class:

$$\frac{d}{d\tau}\int_{\Omega(\tau)} f\,dx = \int_{\Omega(\tau)} \partial_\tau f\,dx + \int_{\partial\Omega(\tau)} f v\cdot\nu\,dS. \tag{1.1}$$

(See the class notes for the notation and precise assumptions.) For simplicity, prove (1.1) in the following particular case. Assume that $n=3$ and that the domains $\Omega(\tau)$ are given by a one-parameter family of one-to-one and onto maps $\varphi=\varphi(\tau,x):\Omega\to\Omega(\tau)=\varphi(\tau,\Omega)$, where $\Omega:=\Omega(0)$ and $\varphi(0,\cdot)=\mathrm{id}_\Omega$, where id_Ω is the identity map on Ω, i.e., $\mathrm{id}_\Omega(x)=x$, $x\in\Omega$.

(a) For each fixed τ, consider the change of variables $x=\varphi(\tau,y)$, so that

$$\int_{\Omega(\tau)} f(\tau,x)\,dx = \int_\Omega f(\tau,\varphi(\tau,y))J(\tau,y)\,dy, \tag{1.2}$$

where $J(\tau,y)$ is the Jacobian of the transformation $x=\varphi(\tau,y)$ for fixed τ.

(b) Show that there exists a on parameter family of vector fields $u(\tau,\cdot)$ such that

$$\partial_\tau\varphi(\tau,x)=u(\tau,\varphi(\tau,x)).$$

(c) Explain why $u=v$ on $\partial\Omega(\tau)$.

(d) Show that

$$\partial_\tau J(\tau,x)=(\mathrm{div}u)(\tau,\varphi(\tau,x))J(\tau,x).$$

(e) Use (1.2) and the above to compute $\frac{d}{d\tau}\int_{\Omega(\tau)} f$, and do an integration by parts to obtain the result.

Problem 2. Let u be a solution to the Cauchy problem for the wave equation in $\mathbb{R}^n$. Assume that u_0 and u_1 have their supports in the ball $B_R(0)$ for some $R>0$. Show that $u=0$ in the exterior of the region

$$I:=\{(t,x)\in(0,\infty)\times\mathbb{R}^n\,|\,x\in B_{R+t}(0)\,\}.$$

I is called a domain of influence for that data on $B_R(0)$ (compare with the $1d$ case).

Problem 3. Let u be a solution to the Cauchy problem for the wave equation and assume that u_0 and u_1 have compact support.

(a) Show that the energy

$$E(t):=\frac{1}{2}\int_{\mathbb{R}^n}\left[(\partial_t u)^2+|\nabla u|^2\right]\,dx$$

is well-defined.

(b) Show that

$$E(t) = E(0),$$

i.e., the energy is conserved.

Problem 4. Let u be a solution to the Cauchy problem for the wave equation in $\mathbb{R}^3$ with compactly supported data (i.e., u_0 and u_1 have compact support).

(a) Show that there exists a constant $C > 0$, depending on u_0 and u_1, such that

$$|u(t,x)| \leq \frac{C}{t}, \tag{1.3}$$

for $t \geq 1$ and $x \in \mathbb{R}^3$. Thus, for each fixed x, u approaches zero as $t \to \infty$, i.e., solutions decay in time.

Hint: Use the formula for solutions in $n = 3$. Since the data has compact support, it vanishes outside $B_R(0)$ for some $R > 0$. This implies an estimate for the area of the largest region within $B_t(x)$ where the data is non-trivial.

(b) Is the estimate (1.3) sharp? (I.e., can it be improved to show that solutions decay faster in time than $\frac{1}{t}$?)

(c) Do we still get decay if the data does not have compact support?

Problem 5. Use Duhamel's principle to show that a solution to the inhomoegeneous wave equation in $1d$ with zero data and source term f is give by

$$u(t,x) = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(t-s,y)\,dyds. \tag{1.4}$$

To do so, first use D'Alembert's formula to conclude that

$$u_s(t,x) = \frac{1}{2}\int_{x-t+s}^{x+t-s} f(s,y)\,dy.$$

Use the definition of u in terms of u_s and change variables to conclude (1.4).

Problem 6. Use Duhamel's principle to show that a solution to the inhomoegeneous wave equation in $3d$ with zero data and source term f is give by

$$u(t,x) = \frac{1}{4\pi}\int_{B_t(x)} \frac{f(t-|y-x|,y)}{|y-x|}\,dy. \tag{1.5}$$

(The integrand in (1.5) is known as the retarded potential.) To do so, first use Kirchhoff's formula for solutions in $n = 3$ to conclude that

$$u_s(t,x) = \frac{t-s}{\mathrm{vol}(\partial B_{t-s}(x))}\int_{\partial B_{t-s}(x)} f(s,y)\,dS(y).$$

Use the definition of u in terms of u_s and change variables to conclude (1.5).

Problem 7. Show that there exists a constant $C > 0$ such that for any solution u to the $3d$ wave equation it holds that

$$|u(t,x)| \leq \frac{C}{t}\int_{\mathbb{R}^3} (|D^2u_0(y)| + |Du_0(y)| + |u_0(y)| + |Du_1(y)| + |u_1(y)|)\,dy$$

for $t \geq 1$.

Hint: Use Kirchhoff's formula, note that for any function f we have

$$f(y) = f(y)\frac{y-x}{t}\cdot\frac{y-x}{t}$$

on $\partial B_t(x)$, and use one of Green's identities.

Problem 8. Consider continuous dependence on the data for the wave equation in $3d$, where smallness on the data part is measured with respect to the norm

$$\|f\|_2 := \int_{\mathbb{R}^3} (|D^2 f(y)| + |Df(y)| + |f(y)|)\, dy.$$

Give a precise formulation of the continuous dependence on the data and prove your statement, i.e., a statement saying that two solutions are close if their corresponding initial data are close.

Hint: Use the estimate of problem 7 as a basis for your statement, and give a similar proof (now you have to also account for $t < 1$).

2. Solutions

Solution 1. (a) This is simply the change of variables formula from calculus.

(b) For each fixed x, the map $\tau \mapsto \varphi(\tau, x)$ is a curve in $\mathbb{R}^3$. $\partial_\tau \varphi(\tau, x)$ is, therefore, the tangent vector to this curve at $\varphi(\tau, x)$ at time τ. The collection of all such tangent vectors, as τ and x vary, forms the vector field u.

(c) The map φ sends $\partial\Omega$ onto $\partial\Omega(\tau)$ for each τ. Since $\partial_\tau\varphi(\tau, x)$ is the velocity at time τ of the particle that started at $x \in \Omega$ at time zero, $u(\tau, \varphi(\tau, x))$ is the velocity of $\partial\Omega(\tau)$ at the point $\varphi(\tau, x) \in \partial\Omega(\tau)$.

(d) According to the notation of part (a), we set

$$\varphi^i_j = \frac{\partial}{\partial y^j}\varphi^i,\ \partial_j u^i = \frac{\partial}{\partial x^j} u^i,$$

where we considered $\varphi = (\varphi^1, \varphi^2, \varphi^3)$. In particular, note that when we write $\varphi^i_j = \partial_j \varphi^i$ the derivative is always with respect to $y \in \Omega$, whereas when we write $\partial_j u^i$ the derivative is always with respect to $x \in \Omega(\tau)$.

Recall the following formula for the determinant of a $n \times n$ matrix a with entries $a^i_j = a^{\text{row}}_{\text{column}}$:

$$\det(a) = \frac{1}{n!}\epsilon_{i_1\cdots i_n}\epsilon^{j_1\cdots j_n} a^{i_1}_{j_1}\cdots a^{i_n}_{j_n}.$$

In our case, this gives

$$J(\tau, y) = \frac{1}{3!}\epsilon_{i_1 i_2 i_3}\epsilon^{j_1 j_2 j_3}\varphi^{i_1}_{j_1}\varphi^{i_2}_{j_2}\varphi^{i_3}_{j_3}.$$

Recall that the definition of J involves an absolute value, which we can omit here since $J > 0$ because $J(0, \cdot) > 0$. Compute

$$\begin{aligned}\partial_\tau \varphi^i_j &= \partial_j \partial_\tau \varphi^i \\ &= \frac{\partial}{\partial y^j} u^i \\ &= \partial_\ell u^i \varphi^\ell_j,\end{aligned}$$

where in the second equality we used (b) and in the third one the chain rule. Therefore

$$\partial_\tau J(\tau, y) = \frac{1}{3!}\epsilon_{i_1 i_2 i_3}\epsilon^{j_1 j_2 j_3}(\partial_\ell u^{i_1}\varphi^\ell_{j_1}\varphi^{i_2}_{j_2}\varphi^{i_3}_{j_3} + \varphi^{i_1}_{j_1}\partial_\ell u^{i_2}\varphi^\ell_{j_2}\varphi^{i_3}_{j_3} + \varphi^{i_1}_{j_1}\varphi^{i_2}_{j_2}\partial_\ell u^{i_3}\varphi^\ell_{j_3}). \tag{2.1}$$

Because $\epsilon_{i_1 i_2 i_3}$ is non-zero only for $i_1 i_2 i_3$ all different from each other, for each triple $i_1 i_2 i_3$, the term $\epsilon_{i_1 i_2 i_3}\partial_\ell u^{i_1}\varphi^\ell_{j_1}\varphi^{i_2}_{j_2}\varphi^{i_3}_{j_3}$ is non-zero only when $\ell = i_1$. Similarly for the second and third terms on the RHS of (2.1), and we obtain

$$\partial_\tau J(\tau, y) = \frac{1}{3!} \sum_{\substack{i_1,i_2,i_3=1 \\ j_1,j_2,j_3=1}}^{3} \epsilon_{i_1 i_2 i_3} \epsilon^{j_1 j_2 j_3} (\partial_{i_1} u^{i_1} + \partial_{i_2} u^{i_2} + \partial_{i_3} u^{i_3}) \varphi^{i_1}_{j_1} \varphi^{i_2}_{j_2} \varphi^{i_3}_{j_3}.$$

Because the summand is non-zero only if $i_1 i_2 i_3$ are all different from each other, the term in parenthesis is always equal to $\partial_1 u^1 + \partial_2 u^2 + \partial_3 u^3 = \mathrm{div} u$, which gives the result.

(e) We have

$$\begin{aligned}
\frac{d}{d\tau} \int_{\Omega(\tau)} f\, dx &= \partial_\tau \int_\Omega f(\tau, \varphi(\tau, y)) J(\tau, y)\, dy \\
&= \int_\Omega \left(\partial_\tau f(\tau, \varphi(\tau, y)) J(\tau, y) + \nabla f(\tau, \varphi(\tau, y)) \cdot \partial_\tau \varphi(\tau, y) J(\tau, y) + f(\tau, \varphi(\tau, y)) \partial_\tau J(\tau, y)\right) dy \\
&= \int_\Omega \big(\partial_\tau f(\tau, \varphi(\tau, y)) J(\tau, y) + \nabla f(\tau, \varphi(\tau, y)) \cdot u(\tau, \varphi(\tau, y)) J(\tau, y) \\
&\qquad + f(\tau, \varphi(\tau, y)) (\mathrm{div} u)(\tau, \varphi(\tau, y)) J(\tau, y)\big)\, dy \\
&= \int_{\Omega(\tau)} \left(\partial_\tau f(\tau, x) + \nabla f(\tau, x) \cdot u(\tau, x) + f(\tau, x)(\mathrm{div} u)(\tau, x)\right) dx \\
&= \int_{\Omega(\tau)} \left(\partial_\tau f(\tau, x) - f(\tau, x)(\mathrm{div} u)(\tau, x) + f(\tau, x)(\mathrm{div} u)(\tau, x)\right) dx \\
&\quad + \int_{\partial\Omega(\tau)} f(\tau, x) u(\tau, x) \cdot \nu(\tau, x)\, dS(x) \\
&= \int_{\Omega(\tau)} \partial_\tau f(\tau, x)\, dx + \int_{\partial\Omega(\tau)} f(\tau, x) v(\tau, x) \cdot \nu(\tau, x)\, dS(x).
\end{aligned}$$

Above, we the steps are as follows: in the second line we used the product rule and the chain rule; in the third line we used (b) and (d); on the fourth line, we changed variables back to x; on the fifth line we integrated ∇f by parts (equivalently, used on of the Green identities); on the last line, we used (c).

Solution 2. Let $(t, x) \notin I$. Then $K^-_{t,x} \cap I = \varnothing$, and the result follows from the finite-propagation speed for the wave equation.

Solution 3. (a) By question 2, the solution u has compact support for each fixed t.

(b) For each t_0 and $\varepsilon > 0$, there exists, by (a), a $R_* > 0$ such that $u(t, x) = 0$ for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ and $|x| \geq R_*$. We now follow the proof of the finite-propagation speed property for the wave equation (see the class notes) using the ball B_{R_*}, and observe the following. In that proof, we did an integration by parts, and controlled the boundary term using the Cauchy-Schwarz inequality. Here, this boundary term vanishes identically by the foregoing. We obtain therefore a sequence of equalities (rather than inequalities as in the proof done in class), which then gives the result.

Solution 4. (a) The solution is given by

$$u(t, x) = \frac{1}{\mathrm{vol}(\partial B_t(x))} \int_{\partial B_t(x)} (u_0(y) + t u_1(y) + \nabla u_0(y) \cdot (y - x))\, dS(y).$$

Since the data is compactly supported, there exists a $R > 0$ such that $u_0(x) = 0$ and $u_1(x) = 0$ for $|x| \geq R$, so that

$$u(t, x) = \frac{1}{\mathrm{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_R(0)} (u_0(y) + t u_1(y) + \nabla u_0(y) \cdot (y - x))\, dS(y).$$

Because the data is compactly supported, we have $|u_0|, |u_1|, |\nabla u_0| \leq C$ for some $C > 0$, so that

$$
\begin{aligned}
|u(t,x)| &\leq \frac{C}{\mathrm{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_R(0)} (1 + t + |y-x|)\, dS \\
&= \frac{C}{\mathrm{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_R(0)} (1 + t + \frac{t|y-x|}{t})\, dS \\
&\leq \frac{C}{\mathrm{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_R(0)} (1 + t + t)\, dS \\
&\leq \frac{C(1+t)}{t^2} \int_{\partial B_t(x) \cap B_R(0)} dS,
\end{aligned}
$$

where we used that $|y-x|/t = 1$ since $y \in B_t(x)$ and that $\mathrm{vol}(\partial B_t(x)) = 4\pi t^2$. Because $\partial B_t(x) \cap B_R(0)$ has area at most $4\pi R^2$, we have the result.

(b) Yes, it cannot be improved for arbitrary solutions of the wave equation. To see this, take $u_0 = 0$ and u_1 to be a non-negative compactly supported function that is equal to 1 on $B_1(0)$. Then

$$
\begin{aligned}
u(t,x) &= \frac{1}{\mathrm{vol}(\partial B_t(x))} \int_{\partial B_t(x)} t u_1(y)\, dS(y) \\
&= \frac{t}{\mathrm{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_1(0)} u_1(y)\, dS(y) + \frac{t}{\mathrm{vol}(\partial B_t(x))} \int_{\partial B_t(x) \setminus (B_t(x) \cap B_1(0))} u_1(y)\, dS(y).
\end{aligned}
$$

Note that the second term on the RHS is always non-negative, thus

$$
u(t,x) \geq \frac{t}{\mathrm{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_1(0)} u_1(y)\, dS(y) = \frac{t}{\mathrm{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_1(0)} dS.
$$

For any x on the boundary of the lightcone, i.e., $|x| = t$, and such that $|x| \geq 1$, we have that the area of $\partial B_t(x) \cap B_1(0)$ is $\geq C > 0$, so that $u(t,x) \geq C/t$.

(c) Not necessarily, e.g., take $u_0 = 0$ and $u_1 = 1$, then $u(t,x) = t$ is the solution.

Solution 5. Using D'Alembert's formula, we find

$$
u_s(t,x) = \frac{1}{2} \int_{x-t+s}^{x+t-s} f(s,y)\, dy,
$$

where we used the fact that D'Alembert's formula was derived for data at $t = 0$; for data at $t = s$ we have to replace t by $t - s$ in the limits of integration. Thus

$$
u(t,x) = \frac{1}{2} \int_0^t \int_{x-t+s}^{x+t-s} f(s,y)\, dyds = \frac{1}{2} \int_0^t \int_{x-z}^{x+z} f(t-z,y)\, dydz,
$$

where we made the change $s = t - z$.

Solution 6. Kirchhoff's formula gives

$$
u_s(t,x) = \frac{1}{\mathrm{vol}(\partial B_{t-s}(x))} \int_{\partial B_{t-s}(x)} (t-s) f(s,y)\, dS(y).
$$

Thus

$$\begin{aligned} u(t,x) &= \int_0^t \frac{t-s}{\operatorname{vol}(\partial B_{t-s}(x))} \int_{\partial B_{t-s}(x)} f(s,y)\, dS(y) ds \\ &= \frac{1}{4\pi} \int_0^t \int_{\partial B_{t-s}(x)} \frac{f(s,y)}{t-s}\, dS(y) ds \\ &= \frac{1}{4\pi} \int_0^t \int_{\partial B_r(x)} \frac{f(t-r,y)}{r}\, dS(y) dr \\ &= \frac{1}{4\pi} \int_{B_t(x)} \frac{f(t-|y-x|,y)}{|y-x|}\, dy, \end{aligned}$$

where we made the change of variables $r = t - s$ and then wrote $r = |y - x|$.

Solution 7. We have

$$u(t,x) = \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} \left(u_0(y) + t u_1(y) + \nabla u_0(y) \cdot (y-x)\right) \, dS(y).$$

The unit outer normal to $\partial B_t(x)$ is $\nu = (y - x)/t$, so that $\nu \cdot \nu = \frac{y-x}{t} \cdot \frac{y-x}{t} = 1$. Therefore, using this and Green's identities,

$$\begin{aligned} \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} u_0(y)\, dS(y) &= \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} u_0(y) \nu \cdot \frac{y-x}{t}\, dS(y) \\ &= \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{B_t(x)} \operatorname{div}_y \left(u_0(y) \frac{y-x}{t} \right) \, dy \\ &= \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{B_t(x)} \left(\nabla u_0(y) \cdot \frac{y-x}{t} + u_0(y) \frac{3}{t} \right) \, dy, \end{aligned}$$

so that

$$\begin{aligned} \left| \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} u_0(y)\, dS(y) \right| &\le \frac{C}{t^2} \int_{B_t(x)} \left(|\nabla u_0(y)| + |u_0(y)|\right) \, dy \\ &\le \frac{C}{t^2} \int_{\mathbb{R}^3} \left(|\nabla u_0(y)| + |u_0(y)|\right) \, dy. \end{aligned}$$

A similar inequality holds for the u_1 integral (with an extra factor of t), and for ∇u_0:

$$\begin{aligned} \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} \nabla u_0(y) \cdot (y-x)\, dS(y) &= \frac{t}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} \nabla u_0(y) \cdot \nu\, dS(y) \\ &= \frac{1}{4\pi t} \int_{B_t(x)} \Delta u_0(y)\, dy, \end{aligned}$$

so that

$$\left| \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} \nabla u_0(y) \cdot (y-x)\, dS(y) \right| \le \frac{C}{t} \int_{\mathbb{R}^3} |D^2 u_0(y)|\, dy.$$

Combining the foregoing produces the result.

Solution 8. We formulate it as follows. Let (u_0, u_1) and (v_0, v_1) be two data sets for the wave equation, and let u and v be the respective solutions. Solutions depend continuously on the data if given $\varepsilon > 0$ and $t > 0$, there exists a $\delta > 0$ such that if

$$\|u_0 - v_0\|_2 + \|u_1 - v_1\|_2 < \delta,$$

then

$$|u(t,x) - v(t,x)| < \varepsilon$$

for all $x \in \mathbb{R}^3$.

We now prove the statement. Set $w_0 = u_0 - v_0$, $w_1 = u_1 - v_1$, and $w = u - v$. By Kirchhoff's formula:

$$w(t,x) = \frac{1}{\mathrm{vol}(\partial B_t(x))} \int_{\partial B_t(x)} (w_0(y) + t w_1(y) + \nabla w_0(y) \cdot (y - x)) \, dS(y).$$

Proceeding as in problem 7, we find

$$\left| \frac{1}{\mathrm{vol}(\partial B_t(x))} \int_{\partial B_t(x)} w_0(y) \, dS(y) \right| \leq \frac{C}{t^2} \int_{\mathbb{R}^3} (|\nabla w_0(y)| + |w_0(y)|) \, dy,$$

$$\left| \frac{1}{\mathrm{vol}(\partial B_t(x))} \int_{\partial B_t(x)} w_1(y) \, dS(y) \right| \leq \frac{C}{t} \int_{\mathbb{R}^3} (|\nabla w_1(y)| + |w_1(y)|) \, dy,$$

and

$$\left| \frac{1}{\mathrm{vol}(\partial B_t(x))} \int_{\partial B_t(x)} \nabla w_0(y) \cdot (y - x) \, dS(y) \right| \leq \frac{C}{t} \int_{\mathbb{R}^3} |D^2 w_0(y)| \, dy.$$

Combining the above we find

$$|w(t,x)| \leq C \max\{\frac{1}{t}, \frac{1}{t^2}\} (\|w_0\|_2 + \|w_1\|_2),$$

which implies the result.

VANDERBILT UNIVERISTY
MATH 8110 — THEORY OF PARTIAL DIFFERENTIAL EQUATIONS
HW 5

Unless stated otherwise, the notation below is as in class.

1. Problems

Problem 1. Let $u \in W^1(\Omega)$. Show that $Du = 0$ a.e. on any set where u is constant.

Problem 2. Is the converse of the previous question true?

Problem 3. Show that $W^{k-1}(\Omega) \subset W^k(\Omega)$.

2. Solutions

Solution 1. This follows from $Du = Du^+ - Du^-$.

Solution 2. Yes, u will be constant a.e. on connected sets. Using the regularization, from $Du = 0$ we have $0 = (Du)_\varepsilon = Du_\varepsilon$ so $u_\varepsilon = c_\varepsilon =$ constant depending on ε. Since $u_\varepsilon \to u$ in $L^1_{loc}(\Omega)$, the numerical constants c_ε converge in the limit $\varepsilon \to 0^+$.

Solution 3. The proof is somewhat technical and long and involves ideas from functional analysis and distribution theory. We will only provide a sketch.

Let $\mathscr{W}^{k,p}(\Omega)$ denote the set of distributions on Ω with the property that all distributional derivatives of order k belong to $L^p(\Omega)$. Let $\Omega' \subset \Omega'' \subset \Omega$ be open sets such that $\operatorname{dist}(\partial\Omega', \partial\Omega'') > \varepsilon$ and $\operatorname{dist}(\partial\Omega'', \partial\Omega) > \varepsilon$ for some fixed $\varepsilon > 0$. Let $\psi \in C_c^\infty(\Omega)$ be such that $\psi = 1$ on Ω''. Let $u \in \mathscr{W}^{k,p}(\Omega)$ and set $T := \psi u$. Take $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}(\varphi) \subset B_\varepsilon(0)$ and $\varphi = 1$ in a neighborhood of the origin.

Consider the polyhamormonic operator Δ^k in $\mathbb{R}^n$, whose fundamental solution is

$$\Gamma(x) = \begin{cases} c|x|^{2k-n}, & \text{for } 2k < n \text{ or for odd } n \text{ such that } n \leq 2k, \\ c|x|^{2k-n} \ln|x|, & \text{for even } n \leq 2k, \end{cases}$$

where c is a constant chosen such that $\Delta^k \Gamma = \delta$, where δ is the Dirac-delta. Then

$$\Delta^k(\varphi\Gamma) = \delta + \zeta$$

where $\zeta \in C_c^\infty(\mathbb{R}^n)$. It follows that

$$T + \zeta * T = \sum_{|\alpha|=k} \frac{k!}{\alpha!} D^\alpha(\varphi\Gamma) * D^\alpha T.$$

Observe that $\zeta * T \in C^\infty(\mathbb{R}^n)$. Using the Leibniz rule for multi-indices we find that over Ω''

$$D^\alpha T = \psi D^\alpha u,$$

so that

$$D^\alpha(\varphi\Gamma) * D^\alpha T = D^\alpha(\varphi\Gamma) * (\psi D^\alpha u)$$

over Ω'. From the theory of singular integrals it follows that the corresponding singular integral operator applied to $\psi D^\alpha u$ is continuous in $L^p(\Omega')$. Thus $u \in L^p_{loc}(\Omega)$, i.e., $\mathscr{W}^{k,p}(\Omega) \subset L^p_{loc}(\Omega)$.

Consider now $u \in L^p_{loc}(\Omega)$, so $u \in L^p(\widetilde{\Omega})$ for any open $\widetilde{\Omega} \subset\subset \Omega$, and let $T = D^\beta u$ be its distributional derivative, $|\beta| = k - \ell$, $1 \le \ell \le k-1$. By assumption $D^\alpha T \in L^p(\widetilde{\Omega})$, $|\alpha| = \ell$, i.e., $T \in \mathscr{W}^{\ell,p}(\widetilde{\Omega})$. By the foregoing $T = D^\beta u \in L^p_{loc}(\Omega)$.

VANDERBILT UNIVERISTY
MATH 8110 — THEORY OF PARTIAL DIFFERENTIAL EQUATIONS
HW 6

Unless stated otherwise, the notation below is as in class.

1. Problems

Problem 1. In class, we proved that any function in $W^{k,p}(\Omega)$ can be approximated by smooth functions up to the boundary if Ω satisfies the segment condition and $1 \leq p < \infty$. It was left as a homework to show that the proof can be reduced to the case of functions with bounded support. Prove this claim.

Problem 2. Prove the change of variables formula stated in class: Let Ω and $\mathcal{D}$ be domains in $\mathbb{R}^n$. Suppose that there exists a one-to-one and onto map $\Psi : \Omega \to \mathcal{D}$ such that $\Psi^j, (\Psi^{-1})^j \in C^k(\Omega)$, have bounded derivatives, $j = 1, \ldots, n$, $k \geq 1$, and $\frac{1}{C} \leq |\det D\Psi| + |\det D\Psi^{-1}| \leq C$ for some constant $C \geq 1$. Given $u \in W^{k,p}(\mathcal{D})$, $1 \leq p < \infty$, define $\tilde{\Psi}(u) : \Omega \to \mathbb{R}$ by $\tilde{\Psi}(u)(x) = u(\Psi(x))$. Then, $\tilde{\Psi}$ transforms $W^{k,p}(\mathcal{D})$ boundedly onto $W^{k,p}(\Omega)$ and has bounded inverse.

2. Solutions

Solution 1. Let $\psi \in C_c^\infty(\mathbb{R}^n)$ satisfy $\psi(x) = 1$ for $|x| \leq 1$, $\psi(x) = 0$ for $|x| \geq 2$, and $|D^\alpha \psi(x)| \leq C$ for $|\alpha| \leq k$. Set $\psi_\varepsilon(x) = \psi(\varepsilon x)$. Then $\psi_\varepsilon(x) = 1$ for $|x| \leq \frac{1}{\varepsilon}$, $\psi(x) = 0$ for $|x| \geq \frac{2}{\varepsilon}$, and $|D^\alpha \psi_\varepsilon(x)| \leq C\varepsilon^{|\alpha|} \leq C$ for $|\alpha| \leq k$ and $0 < \varepsilon \leq 1$. If $u \in W^{k,p}(\Omega)$, then $u_\varepsilon := \psi_\varepsilon u$ belongs to $W^{k,p}(\Omega)$, has bounded support, and

$$|D^\alpha u_\varepsilon| \leq C \sum_{\beta \leq \alpha} |D^\beta u D^{\alpha-\beta} \psi_\varepsilon| \leq \sum_{\beta \leq \alpha} |D^\beta u|.$$

Set $\Omega_\varepsilon := \{x \in \Omega \,|\, |x| > \frac{1}{\varepsilon}\}$. Since $u - u_\varepsilon = (1 - \psi_\varepsilon)u = 0$ for $|x| \leq \frac{1}{\varepsilon}$, we have

$$\|u - u_\varepsilon\|_{W^{k,p}(\Omega)} = \|u - u_\varepsilon\|_{W^{k,p}(\Omega_\varepsilon)} \leq \|u\|_{W^{k,p}(\Omega_\varepsilon)} + \|u_\varepsilon\|_{W^{k,p}(\Omega_\varepsilon)} \leq C\|u\|_{W^{k,p}(\Omega_\varepsilon)}$$

which goes to zero when $\varepsilon \to 0$.

Solution 2. The map $\tilde{\Psi}$ is well-defined for a.e. functions since $k \geq 1$. Let $\{u_j\}$ be a sequence of smooth functions converging to u in $W^{k,p}(\Omega)$. Let $|\alpha| \leq k$. Successive applications of the chain rule and the product rule give

$$D^\alpha \tilde{\Psi}(u_j)(x) = \sum_{\beta \leq \alpha} p_{\alpha\beta}(x) D_y^\beta u_j(y) = \sum_{\beta \leq \alpha} p_{\alpha\beta}(x) \tilde{\Psi}(D^\beta u_j)(x),$$

where $y = \Psi(x)$ and $p_{\alpha\beta}$ is a polynomial of degree $\leq |\beta|$ in derivatives of Ψ^j of order $\leq |\alpha|$, $j = 1, \ldots, n$.

Let $\varphi \in C_c^\infty(\Omega)$. We have

$$(-1)^{|\alpha|} \int_\Omega \tilde{\Psi}(u_j)(x) D^\alpha \varphi(x)\, dx = \sum_{\beta \leq \alpha} \int_\Omega p_{\alpha\beta}(x) \tilde{\Psi}(D^\beta u_j)(x) \varphi(x)\, dx.$$

But

$$(-1)^{|\alpha|} \int_\Omega \tilde{\Psi}(u_j)(x) D^\alpha \varphi(x)\, dx = \int_{\mathcal{D}} \underbrace{\tilde{\Psi}(u_j)(\Psi^{-1}(y))}_{=u_j(y)} (D^\alpha \varphi)(\Psi^{-1}(y)) |\det D\Psi^{-1}(y)|\, dy$$

and

$$\sum_{\beta\leq\alpha}\int_\Omega p_{\alpha\beta}(x)\tilde{\Psi}(D^\beta u_j)(x)\varphi(x)\,dx=\sum_{\alpha\leq\beta}\int_{\mathcal{D}} p_{\alpha\beta}(\Psi^{-1}(y))\underbrace{\tilde{\Psi}(D^\beta u_j)(\Psi^{-1}(y))}_{=D^\beta u_j(y)}\varphi(\Psi^{-1}(y))|\det D\Psi^{-1}(y)|\,dy.$$

Since $D^\beta u_j \to u$ in L^p, we can replace u_j by u above and change variables back to get

$$(-1)^{|\alpha|}\int_\Omega \tilde{\Psi}(u)(x)D^\alpha\varphi(x)\,dx=\sum_{\beta\leq\alpha}\int_\Omega p_{\alpha\beta}(x)\tilde{\Psi}(D^\beta u)(x)\varphi(x)\,dx.$$

Thus, $\tilde{\Psi}(u)\in W^{k,p}(\Omega)$ and

$$D^\alpha\tilde{\Psi}(u)(x)=\sum_{\beta\leq\alpha}p_{\alpha\beta}(x)\tilde{\Psi}(D^\beta u)(x).$$

Then

$$\begin{aligned}\int_\Omega |D^\alpha\tilde{\Psi}(u)(x)|^p\,dx &\leq C\max_{|\beta|\leq|\alpha|}\sup_{x\in\Omega}|p_{\alpha\beta}(x)|^p\int_\Omega \underbrace{|D^\alpha\tilde{\Psi}(u)(x)|^p}_{=|(D^\beta u)(\Psi(x))|^p}\,dx\\ &\leq C\max_{|\beta\leq|\alpha|}\int_{\mathcal{D}}|D^\beta u(y)|^p|\det D\Psi^{-1}(y)|\,dy\\ &\leq C\|u\|_{W^{k,p}(\mathcal{D})},\end{aligned}$$

thus $\|\tilde{\Psi}(u)\|_{W^{k,p}(\Omega)}\leq C\|u\|_{W^{k,p}(\mathcal{D})}$. Repeating the argument with Ψ^{-1} in place of Ψ gives the result.

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HW 7

Unless stated otherwise, the notation below is as in class.

1. Problems

Problem 1. Prove the following lemma stated (but not proven) in class: Let $p > 1$, $kp < n$, $p^* = \frac{np}{n-kp}$. There exist a constant $K > 0$ such that

$$\|\chi_1 * |u|\|_{L^{p^*}(\mathbb{R}^n)} \leq \|\chi_1 G_k * |u|\|_{L^{p^*}(\mathbb{R}^n)} \leq \|G_k * |u|\|_{L^p(\mathbb{R}^n)} \leq K\|u\|_{L^p(\mathbb{R}^n)}$$

for all $u \in L^p(\mathbb{R}^n)$.
Hint: Adapt the ideas of the proof given in class of a similar, albeit simpler, inequality.

Problem 2. Prove that $u \in W_0^{1,p}(\Omega)$ ($1 \leq p < \infty$, $\partial\Omega$ a C^1 boundary, Ω bounded), if and only if $Tu = 0$, where T is the trace operator.

Problem 3. Prove the uniqueness statement in the proof of the "Riesz representation for Sobolev spaces" (the part that was not done in class).

2. Solutions

Solution 1. All inequalities are a direct consequence of the definitions but the last one. Using Hölder's inequality we have

$$\begin{aligned}\int_{\mathbb{R}^n \setminus B_r(x)} |u(y)||x-y|^{k-n}\,dy &\leq \|u\|_{L^p(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n \setminus B_r(x)} |x-y|^{(k-n)p'}\,dy\right)^{\frac{1}{p'}} \\ &\leq C\|u\|_{L^p(\mathbb{R}^n)} \left(\int_r^\infty t^{(k-n)p'+n-1}\,dt\right)^{\frac{1}{p'}} \\ &\leq C_1 r^{k-\frac{n}{p}} \|u\|_{L^p(\mathbb{R}^n)},\end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and we used that $kp < n$.

For $\tau > 0$, let r be such that $C_1 r^{k-\frac{n}{p}} \|u\|_{L^p(\mathbb{R}^n)} = \frac{\tau}{2}$. If

$$G_k * |u|(x) = \int_{\mathbb{R}^n} |u(y)||x-y|^{k-n}\,dy > \tau,$$

then

$$\chi_r G_k * |u|(x) = \int_{B_r(x)} |u(y)||x-y|^{k-n}\,dy > \frac{\tau}{2}. \tag{2.1}$$

Thus,

$$
\begin{aligned}
|\{x \mid G_k * |u|(x) > \tau\}| &\le \left|\{x \mid \chi_r G_k * |u|(x) > \frac{\tau}{2}\}\right| \\
&\le \left(\frac{2}{\tau}\right)^p \|\chi_r G_k * |u|\|^p_{L^p(\mathbb{R}^n)} \\
&\le \left(\frac{r^{\frac{n}{p}-k}}{C_1\|u\|_{L^p(\mathbb{R}^n)}}\right)^p C r^{kp}\|u\|^p_{L^p(\mathbb{R}^n)} \\
&= C_2 r^n,
\end{aligned}
$$

where we used the similar lemma about convolutions proved in class. Since

$$
r^n = \left(\frac{2C_1}{\tau}\|u\|_{L^p(\mathbb{R}^n)}\right)^{p^*}
$$

we have that

$$
|\{x \mid G_k * |u|(x) > \tau\}| \le C_2 \left(\frac{2C_1}{\tau}\|u\|_{L^p(\mathbb{R}^n)}\right)^{p^*}.
$$

Therefore, the map

$$
u \mapsto G_k * |u|
$$

is of weak type (p, p^*).

The values of p satisfying the our assumptions form an open set, so we can find p_1 and p_2 in that set and $0 < \theta < 1$ such that

$$
\begin{aligned}
\frac{1}{p} &= \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \\
\frac{1}{p^*} &= \frac{1}{p} - \frac{k}{n} = \frac{1-\theta}{p_1^*} + \frac{\theta}{p_2^*}.
\end{aligned}
$$

Because $p^* > p$, the Marcinkiewicz interpolation theorem implies that the map $u \mapsto G_k * |u|$ is bounded from $L^p(\mathbb{R}^n)$ to $L^{p^*}(\mathbb{R}^n)$.

Solution 2. Since the trace is continuous from $W^{1,p}(\Omega)$ to $W^{1,p}(\partial\Omega)$ and every element in $C_c^\infty(\Omega)$ has zero trace, we conclude that $Tu = 0$ for $u \in W_0^{1,p}(\Omega)$.

Suppose now that $u \in W^{1,p}(\Omega)$ satisfies $Tu = 0$. As usual, we can reduce the proof to the case $\Omega = \{x^n > 0\}$. Extend u to be zero outside Ω and denote $\tilde{u}$ this extension. Let $u_j \in C^\infty(\overline{\Omega})$ be a sequence converging to u in $W^{1,p}(\Omega)$. The difference

$$
\int_{\mathbb{R}^n} \tilde{u}_j D^\alpha \varphi - (-1)\int_{\mathbb{R}^n} \widetilde{D^\alpha u_j}\varphi
$$

is a sum of integrals of the form

$$
\int_{\mathbb{R}^{n-1}} u_j(x^1, \dots, x^{n-1}, 0)\varphi(x^1, \dots, x^{n-1}, 0)
$$

which tends to zero by the assumption of zero trace on u. Hence,

$$
\int_{\mathbb{R}^n} \tilde{u} D^\alpha \varphi - (-1)\int_{\mathbb{R}^n} \widetilde{D^\alpha u}\varphi = 0
$$

and $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$. The result now follows from the following claim, which we prove below: $u \in W_0^{k,p}(\Omega)$ if and only if the zero extension of u belongs to $W^{k,p}(\mathbb{R}^n)$.

It is not difficult to see that if $u \in W_0^{k,p}(\Omega)$ then $\tilde{u} \in W^{k,p}(\mathbb{R}^n)$. To prove the converse, we argue as in the proof of approximation of Sobolev functions by smooth functions up to the boundary, producing the functions u_j. Translate $\tilde{u}_j$ by $\tilde{u}_{j,t}(x) = \tilde{u}_j(x - ty)$, where y is as in that proof (but there we translated by $+$ whereas here we translate by $-$). The translation $x - ty$ moves the support of $\tilde{u}_j$ to inside Ω so $u_{j,t}$ belongs to $W^{k,p}(\mathbb{R}^n)$ since $\tilde{u}_{j,t}$ does. The restriction of $u_{j,t}$ to Ω belongs to $W_0^{k,p}(\Omega)$ since $u_{j,t}$ vanishes outside a compact subset of Ω and these restrictions converge to u_j as $t \to 0^+$.

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HW 8

Unless stated otherwise, the notation below is as in class.

1. Problems

Problem 1. Prove the uniqueness statement in the proof of the "Riesz representation for Sobolev spaces" (the part that was not done in class).

Problem 2. Prove that

$$\|D^{\alpha_1}u_i \cdots D^{\alpha_\ell}u_\ell\|_{L^2(\mathbb{R}^n)} \leq C\sum_{i=1}^{\ell}\|D^k u_i\|_{L^2(\mathbb{R}^n)}\prod_{j\neq i}\|u_j\|_{L^\infty(\mathbb{R}^n)},$$

for $u_i \in H^k(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ and $\sum_i |\alpha_i| = k$.
Hint: You can use, without proof, the Gagliardo-Nirenberg inequality

$$\|D^j u\|_{L^{\frac{2r}{j}}(\mathbb{R}^n)} \leq C\|u\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{j}{r}}\|D^r u\|_{L^2(\mathbb{R}^n)}^{\frac{j}{r}}.$$

Problem 3. In the context of Egorov's example, prove the lemma that reduces the necessary condition for existence of weak solutions to

$$\|v\|_0 \leq C\|L^* v\|_N,$$

$v \in C_c^\infty(\Omega)$.

2. Solutions

Solution 1. We follow the notation used in class. Suppose the conclusion holds for v_1 and v_2 attaining the minimum, so

$$\|v_1\|_{L^{p'}(\Omega_{(k)})} = \|f\|_{(W^{k,p}(\Omega))'} = \|v_2\|_{L^{p'}(\Omega_{(k)})} = 1,$$

where we can assume $= 1$ upon redefining f as $\frac{f}{\|f\|_{(W^{k,p}(\Omega))'}}$, and for all $u \in W^{k,p}(\Omega)$,

$$f(u) = \sum_{|\alpha|\leq k}\langle v_1, D^\alpha u\rangle = \sum_{|\alpha|\leq k}\langle v_2, D^\alpha u\rangle.$$

First, we claim that there exists a unique $x \in X$ such that

$$f^*(x) = \|x\|_{L^p(\Omega_{(k)})} = 1.$$

Since $\|f\|_{(W^{k,p}(\Omega))'} = \|f^*\|_{X'} = 1$, there exists $\{x_i\} \subset X$ such that $\|x_i\|_{L^p(\Omega_{(k)})} = 1$ and $|f^*(x_i)| \to 1$; we can further assume that $f^*(x_i) \to 1$ by modifying the sequence if necessary. Because $L^p(\Omega_{(k)})$ is uniformly convex for $1 < p < \infty$, given $0 < \varepsilon \leq 2$, there exists a $\delta > 0$ such that if $\|x_i - x_j\|_{L^p(\Omega_{(k)})} \geq \varepsilon$ then $\|\frac{x_i+x_j}{2}\|_{L^p(\Omega_{(k)})} \leq 1-\delta$, thus if $\|\frac{x_i+x_j}{2}\|_{L^p(\Omega_{(k)})} > 1-\delta$ we must have $\|x_i - x_j\|_{L^p(\Omega_{(k)})} < \varepsilon$. For large i we have $f^*(x_i) > 1-\delta$ thus for large i,j we also have $f^*(\frac{x_i+x_j}{2}) > 1-\delta$. Hence, as f^* is continuous with norm 1, $1-\delta < f^*(\frac{x_i+x_j}{2}) \leq \|\frac{x_i+x_j}{2}\|_{L^p(\Omega_{(k)})}$. Therefore, $\|x_i - x_j\|_{L^p(\Omega_{(k)})} < \varepsilon$ and $\{x_i\}$ is Cauchy, thus $x_i \to x$ in $L^p(\Omega_{(k)})$ and $x \in X$ since X is closed. Clearly $\|x\|_{L^p(\Omega_{(k)})} = 1$

and $f^*(x) = 1$. To obtain uniqueness, if there are two such x's, say, x_1 and x_2, we can apply the above argument to the sequence $\{x_1, x_2, x_1, x_2, \dots\}$, which must converge.

Since v_1 and v_2 are two representatives of f^*, we have

$$f^*(x) = 1 = \sum_{|\alpha|\leq k} \langle (v_1)_\alpha, x_\alpha\rangle = \sum_{|\alpha|\leq k} \langle (v_2)_\alpha, x_\alpha\rangle.$$

Consider the following claim: given $w \in L^p(\Omega_{(k)})$ with $\|w\|_{L^p(\Omega_{(k)})} = 1$, there exists at most one $\ell \in (L^p(\Omega_{(k)}))'$ such that $\|\ell\|_{(L^p(\Omega_{(k)}))'} = 1$ and $\ell(w) = 1$.

Let $\tilde{v}_1$ and $\tilde{v}_2$ be the extensions of v_1 and v_2, considered as linear functionals on X, to $L^p(\Omega_{(k)})$ given by Hahn-Banach. Thus $\|\tilde{v}_1\|_{(L^p(\Omega_{(k)}))'} = 1 = \|\tilde{v}_2\|_{(L^p(\Omega_{(k)}))'}$ (observe taht even though $\tilde{v}_1 = f^* = \tilde{v}_2$ on X, we cannot claim from this that $\tilde{v}_1 = \tilde{v}_2$ because the Hanh-Banach extensions might not be unique), and by the foregoing we have $\tilde{v}_1(x) = 1 = \tilde{v}_2(x)$. Thus $\tilde{v}_1 = \tilde{v}_2$ by the above claim.

It remains to prove the above claim. Suppose that there are two such $\ell's$, ℓ_1 and ℓ_2, $\ell_1 \neq \ell_2$. Thus $\ell_1(u) \neq \ell_2(u)$ for some $u \in L^p(\Omega_{(k)})$. We can assume that $\ell_1(u) - \ell_2(u) = 2$ upon replacing u by a suitable multiple of itself, and that $\ell_1(u) = 1$ and $\ell_2(u) = -1$ upon replacing u with its sum with a suitable multiple of w. Thus

$$\begin{aligned}\ell_1(w + tu) &= 1 + t,\\ \ell_2(w - tu) &= 1 + t,\end{aligned}$$

$t > 0$. Since $\|\ell_1\|_{(L^p(\Omega_{(k)}))'} = 1 = \|\ell_2\|_{(L^p(\Omega_{(k)}))'}$,

$$\begin{aligned}1 + t = \ell_1(w + tu) &\leq \|w + tu\|_{L^p(\Omega_{(k)})},\\ 1 + t = \ell_1(w - tu) &\leq \|w + tu\|_{L^p(\Omega_{(k)})}.\end{aligned}$$

Recall the L^p-parallelogram inequalities:

$$\begin{aligned}&\|\frac{a+b}{2}\|_{L^p}^p + \|\frac{a-b}{2}\|_{L^p}^p \geq \frac{1}{2}\|a\|_{L^p}^p + \frac{1}{2}\|b\|_{L^p}^p,\ 1 < p \leq 2,\\ &\|\frac{a+b}{2}\|_{L^p}^{p'} + \|\frac{a-b}{2}\|_{L^p}^{p'} \geq (\frac{1}{2}\|a\|_{L^p}^p + \frac{1}{2}\|b\|_{L^p}^p)^{p'-1},\ 2 \leq p < \infty.\end{aligned}$$

If $1 < p \leq 2$, we get

$$\begin{aligned}1 + t^p\|u\|_{L^p(\Omega_{(k)})}^p &= \|\frac{(w+tu)+(w-tu)}{2}\|_{L^p(\Omega_{(k)})}^p + \|\frac{(w+tu)-(w-tu)}{2}\|_{L^p(\Omega_{(k)})}^p\\ &\geq \frac{1}{2}\|w + tu\|_{L^p(\Omega_{(k)})}^p + \frac{1}{2}\|w - tu\|_{L^p(\Omega_{(k)})}^p\\ &\geq (1+t)^p,\end{aligned}$$

which cannot be true for all $t > 0$. If $2 \leq p \leq \infty$, we apply the second inequality to get

$$\begin{aligned}1 + t^{p'}\|u\|_{L^p(\Omega_{(k)})}^{p'} &= \|\frac{(w+tu)+(w-tu)}{2}\|_{L^p(\Omega_{(k)})}^{p'} + \|\frac{(w+tu)-(w-tu)}{2}\|_{L^p(\Omega_{(k)})}^{p'}\\ &\geq (\frac{1}{2}\|w + tu\|_{L^p(\Omega_{(k)})}^p + \frac{1}{2}\|w - tu\|_{L^p(\Omega_{(k)})}^p)^{p'-1}\\ &\geq (1+t)^{p'},\end{aligned}$$

which again is an impossibility.

Solution 2. From Hölder's inequality and the product rule,

$$\begin{aligned}\|D^\alpha(uv)\|_{L^2(\mathbb{R}^n)} &\leq \sum_{\beta\leq\alpha} C\|D^\beta u D^{\alpha-\beta} v\|_{L^2(\mathbb{R}^n)}\\ &\leq C\sum_{\beta\leq\alpha} \|D^\beta u\|_{L^{\frac{2k}{|\beta|}}(\mathbb{R}^n)} \|D^{\alpha-\beta} v\|_{L^{\frac{2k}{|\alpha-\beta|}}(\mathbb{R}^n)}\end{aligned}$$

The Gagliardo-Niremberg inequality gives

$$\begin{aligned}
\|D^\alpha(uv)\|_{L^2(\mathbb{R}^n)} &\leq C\sum_{\beta\leq\alpha}\|u\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{|\beta|}{k}}\|D^k u\|_{L^2(\mathbb{R}^n)}^{\frac{|\beta|}{k}}\|v\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{|\alpha-\beta|}{k}}\|D^k v\|_{L^2(\mathbb{R}^n)}^{\frac{|\alpha-\beta|}{k}} \\
&\leq C\sum_{\beta\leq\alpha}\left(\|u\|_{L^\infty(\mathbb{R}^n)}\|D^k v\|_{L^2(\mathbb{R}^n)}\right)^{\frac{|\alpha-\beta|}{k}}\left(\|D^k u\|_{L^2(\mathbb{R}^n)}\|v\|_{L^\infty(\mathbb{R}^n)}\right)^{\frac{|\beta|}{k}} \\
&\leq C(\|u\|_{L^\infty(\mathbb{R}^n)}\|D^k v\|_{L^2(\mathbb{R}^n)}+\|v\|_{L^\infty(\mathbb{R}^n)}\|D^k u\|_{L^2(\mathbb{R}^n)})
\end{aligned}$$

which implies the result.

Solution 3. Using that we now established $H-k(\Omega)\approx(H^k(\Omega))'$ for $k\in\mathbb{Z}$ (this had been established initially for $k\geq 0$), the necessary condition for existence be be extended for $s,t\in\mathbb{Z}$. Thus, there exist $s,t\in\mathbb{Z}$ such that

$$\|v\|_s\leq C\|L^*v\|_t.$$

If $s\geq 0$, then we can choose $N\geq t$. Otherwise, we can assume $t\geq s$ since if $t<s$ then we can choose $\tilde{t}\geq s$ and work with $\tilde{t}$ (since $\|L^*v\|_t\leq\|L^*v\|_{\tilde{t}}$ then). Because $D_x^\alpha v\in C_c^\infty(\Omega)$ if $v\in C_c^\infty(\Omega)$, we can apply the inequality to $D_x^\alpha v$ to get

$$\|D_x^\alpha v\|_s\leq C\|L^*D_x^\alpha v\|_t\leq C\|D_x^\alpha L^*v\|_t\leq C\|D_x^\alpha L^*v\|_{t+|\alpha|},$$

where we used that $L^*v=\partial_t^2 v-a(t)\partial_x^2 v-b(t)\partial_x v$. We also have

$$\|\partial_t^2 v\|_{s-1}\leq C(\|L^*v\|_{s-1}+\|\partial_x^2 v\|_{s-1}+\|\partial_x v\|_{s-1})\leq\|L^*v\|_{t+1},$$

where we used $\|L^*v\|_{s-1}\leq\|L^*v\|_{t+1}$ by $s\leq t$ and $\|\partial_x^2 v\|_{s-1}+\|\partial_x v\|_{s-1}\leq\|L^*v\|_{t+1}$ by the above. Then

$$\begin{aligned}
\|v\|_{s+1}\leq& C(\|v\|_s+\|\partial_t^2 v\|_{s-1}+\|\partial_x^2 v\|_{s-1}) \\
&\leq C(\|L^*v\|_t+\|L^*v\|_{t+1}) \\
&\leq\|L^*v\|_{t+1}.
\end{aligned}$$

Iterating this argument gives the result.

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HW 9

Unless stated otherwise, the notation below is as in class.

1. Problems

Problem 1. Prove the following statement. Let $Lu \geq f\,(= f)$ in a bounded domain Ω, $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, and assume that $c \leq 0$. Then, there exists a constant $C > 0$ depending only on the diameter of Ω and on $\frac{\|b\|_{L^\infty(\Omega)}}{\Lambda}$ sub that

$$\sup_{\Omega} u\,(|u|) \leq \sup_{\partial\Omega} u^+\,(|u|) + C \sup_{\Omega} \frac{|f^-|}{\Lambda}\,\left(\frac{|f|}{\Lambda}\right).$$

$(f^- = \inf\{f, 0\}, u^+ = \sup\{u, 0\}.)$

Problem 2. Prove the following statement. Let $Lu = f$ in a bounded domain Ω, $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, and assume that $c \leq 0$. Let C be the constant of the previous problem and suppose that

$$A = 1 - C \sup_{\Omega} \frac{c^+}{\Lambda} > 0.$$

Then

$$\sup_{\Omega} |u| \leq \frac{1}{A}\left(\sup_{\Omega} |u| + C \sup_{\Omega} \frac{|f|}{\Lambda}\right).$$

2. Solutions

Solution 1. Let Ω lie in the slap $0 < x^1 < d$ and set $L_0 = a^{ij}\partial_i\partial_j + b^i\partial_i$. If $\alpha > \frac{\|b\|_{L^\infty(\Omega)}}{\Lambda} + 1$, then

$$\begin{aligned} L_0 e^{\alpha x^1} &= (\alpha^2 a^{11} + \alpha b^1)e^{\alpha x^1} \\ &\geq (\alpha^2\Lambda - \alpha\|b\|_{L^\infty(\Omega)})e^{\alpha x^1} \\ &= (\alpha^2\Lambda - \alpha\Lambda\frac{\|b\|_{L^\infty(\Omega)}}{\Lambda})e^{\alpha x^1} \\ &\geq \Lambda. \end{aligned}$$

Set

$$v = \sup_{\partial\Omega} u^+ + (e^{\alpha d} - e^{\alpha x^1}) \sup_{\Omega} \frac{|f^-|}{\Lambda} \geq 0.$$

Then

$$\begin{aligned} Lv &= -(L_0 e^{\alpha x^1}) \sup_{\Omega} \frac{|f^-|}{\Lambda} + cv \\ &\leq -\sup_{\Omega} |f^-|, \end{aligned}$$

thus

$$L(v - u) \leq -\sup_{\Omega} |f^-| - f \leq 0.$$

We also have $v - u \geq 0$ on $\partial\Omega$. Thus, by one of the corollaries of the maximum principle, $u \leq v$, so

$$\begin{aligned} u &\leq \sup_{\partial\Omega} u^+ + (e^{\alpha d} - e^{\alpha x^1}) \sup_{\Omega} \frac{|f^-|}{\Lambda} \\ &\leq \sup_{\partial\Omega} u^+ + (e^{\alpha d} - 1) \sup_{\Omega} \frac{|f^-|}{\Lambda}. \end{aligned}$$

Solution 2. Write $Lu = (L_0 + c)u = f$ as $(L_0 + c^-)u = f - c^+ u =: \tilde{f}$. From the previous problem,

$$\begin{aligned} \sup_{\Omega} |u| &\leq \sup_{\partial\Omega} |u| + C \sup_{\Omega} \frac{|\tilde{f}|}{\Lambda} \\ &\leq \sup_{\partial\Omega} |u| + C \left(\sup_{\Omega} \frac{|f|}{\Lambda} + \sup_{\Omega} |u| \sup_{\Omega} \frac{|c^+|}{\Lambda} \right). \end{aligned}$$

Thus

$$\left(1 - C \sup_{\Omega} \frac{|c^+|}{\Lambda} \right) \sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + C \sup_{\Omega} \frac{|f|}{\Lambda}.$$

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HW 10

Unless stated otherwise, the notation below is as in class.

1. Problems

Problem 1. Prove the following statement used in the proof of existence of solutions to linear first-order symmetric hyperbolic systems. Let $u \in L^\infty\left(J, H^k(\mathbb{R}^n)\right)$, where k is a non-negative integer and J is an open interval. Then there exists a $\tilde{u} \in L^2_{loc}(J \times \mathbb{R}^n)$ that is k times weakly differentiable with respect to x and with derivatives in $L^2_{loc}(J \times \mathbb{R}^n)$ and such that

$$\langle \varphi, u \rangle = \int_{J \times \mathbb{R}^n} \varphi \tilde{u} \, dx \, dt,$$

for all $\varphi \in C_c^\infty(J \times \mathbb{R}^n)$.

2. Solutions

Solution 1. Consider first $k = 0$. Let u_i be a sequence of step functions converging point-wise a.e. to u. Thus

$$u_i = \sum_{\ell=1}^{N_i} f_{i,\ell} \chi_{A_{i,\ell}},$$

where $A_{i,\ell} \subset J$ are measurable sets and $f_{i,\ell} \in L^2(\mathbb{R}^n)$. u_ℓ defines a $dt \times dx$-measurable function which is also measurable in $J \times \mathbb{R}^n$ (i.e., measurable with respect to the $(n+1)$-dimensional Lebesgue measure, since the $(n+1)$-dimensional Lebesgue measure is the completion of the $1 \times n$ measure).

Let $B_i := \{t \in J \mid \|u_i(t)\|_{L^2(\mathbb{R}^n)} \leq 2\|u(t)\|_{L^2(\mathbb{R}^n)}\}$. Set $u_i' = \chi_{B_i} u_i$. Then, u_i' converges dt-a.e. to u with respect to L^2 and is measurable in $J \times \mathbb{R}^n$. Given a compact set $K \subset J \times \mathbb{R}^n$, let K_1 be its projection onto J and set $U_i = u_i' \chi_K$; U_i is $dt \times dx$-measurable. We have

$$\left(\int_{\mathbb{R}^n} |U_i(t,x) - U_j(t,x)| \, dx \right)^{\frac{1}{2}} \leq \chi_{K_1}(t) \|u_i'(t) - u_j'(t)\|_{L^2(\mathbb{R}^n)}.$$

Consider

$$\|u_i'(t) - u_j'(t)\|_{L^2(\mathbb{R}^n)} \leq \|u_i'(t) - u(t)\|_{L^2(\mathbb{R}^n)} + \|u(t) - u_j'(t)\|_{L^2(\mathbb{R}^n)}.$$

Each term on the RHS converges to zero point-wise a.e. (in t) and is bounded by a function in $L^\infty(J)$; thus, dominated convergence implies that $\{U_i\}$ is a Cauchy sequence in $L^2(K)$. Thus, for each K we have an element $U_K \in L^2(J \times \mathbb{R}^n)$. Taking an increasing sequence of compact sets we obtain a locally square-integrable function U in $J \times \mathbb{R}^n$. We finally observe that

$$\langle \varphi, u \rangle = \lim_{i \to \infty} \langle \varphi, u_i' \rangle = \int_{J \times \mathbb{R}^n} \varphi U \, dt \, dx$$

for all $\varphi \in C_c^\infty(J \times \mathbb{R}^n)$. This gives the result for $k = 0$.

For $k \geq 1$, we apply the above to the function $D^{\vec{\alpha}} u \in L^\infty\left(J, L^2(\mathbb{R}^n)\right)$ to obtain U_α such that

$$\langle \varphi, D^{\vec{\alpha}} u \rangle = \int_{J \times \mathbb{R}^n} \varphi U_\alpha \, dt \, dx$$

for all $\varphi \in C_c^\infty(J \times \mathbb{R}^n)$, which gives the result.

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MATH 8110 — THEORY OF PARTIAL DIFFERENTIAL EQUATIONS
HW 11

Unless stated otherwise, the notation below is as in class.

1. Problems

Problem 1. In class, we defined the concept of a function with local compact support in x, and discussed that a smooth function $f\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^d$ can be regarded as an element of $C^m\left(\mathbb{R}, H^k(\mathbb{R}^n, \mathbb{R}^d)\right)$ for any $m, k \geq 0$. Show that this is not the case if f is assumed only to be such that for each fixed t, $f(t,\cdot)$ has compact support.

Hint: Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ and define f by

$$f(t,x) = \begin{cases} \varphi(x^1 - \frac{1}{t}, x^2, \dots, x^n), & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Problem 2. Verify the inequalities $\mathcal{M}_k[v_0] \leq \mathcal{C}$ and $\mathcal{M}_k[v_1] \leq \mathcal{C}$ in the proof of local existence and uniqueness of solutions to quasilinear wave equations.

Problem 3. Let $\{f_i\} \subset H^k(\mathbb{R}^n)$ be a bounded sequence that converges to f in $H^\ell(\mathbb{R}^n)$, $\ell < k$. Show that $f \in H^k(\mathbb{R}^n)$.

2. Solutions

Solution 1. Observe that f is smooth for $t > 0$ and for $t < 0$. For each $(0,x)$, there exists a neighborhood U of $(0,x)$ in $\mathbb{R}\times\mathbb{R}^n$ such that $f = 0$ in U. Thus, f is smooth. For fixed t, $f(t,\cdot)$ has compact support. For $t \leq 0$, $\|f(t,\cdot)\|_{L^2(\mathbb{R}^n)} = 0$. But for $t > 0$, $\|f(t,\cdot)\|_{L^2(\mathbb{R}^n)} > 0$. Thus $f \notin C^0\left(\mathbb{R}, H^0(\mathbb{R}^n,\mathbb{R})\right)$.

Solution 2. We have $\mathcal{M}_k[v_0] = \mathcal{M}_k[u_{0,0}] \leq C_0+1$ by assumption, so we can choose $\mathcal{C} \geq C_0+1$. For v_1, we need $\mathcal{N}[v_{i-1}] = \mathcal{N}[v_0] \leq z_I(\mathcal{C})$. In the proof, this was obtained using the induction hypothesis for v_{i-2}, which would give v_{-1} here, which has not been defined. But we have $\mathcal{N}[v_0] \leq z_I(\mathcal{C})$ directly from the fact that v_0 is constant in time and from Sobolev embedding.

Solution 3. Since the sequence is bounded in $H^k(\mathbb{R}^n)$ it converges weakly to a limit $\tilde{f} \in H^k(\mathbb{R}^n)$. Because $H^k(\mathbb{R}^n) \hookrightarrow H^\ell(\mathbb{R}^n)$ compactly, f_i converges to $\tilde{f}$ in $H^\ell(\mathbb{R}^n)$. Uniqueness of the limit gives $\tilde{f} = f$.