VANDERBILT UNIVERISTY
MATH 8110 — THEORY OF PARTIAL DIFFERENTIAL EQUATIONS
HW 7

Unless stated otherwise, the notation below is as in class.

1. Problems

Problem 1. Prove the following lemma stated (but not proven) in class: Let $p > 1$, $kp < n$, $p^* = \frac{np}{n-kp}$. There exist a constant $K > 0$ such that

$$\|\chi_1 * |u|\|_{L^{p^*}(\mathbb{R}^n)} \leq \|\chi_1 G_k * |u|\|_{L^{p^*}(\mathbb{R}^n)} \leq \|G_k * |u|\|_{L^p(\mathbb{R}^n)} \leq K\|u\|_{L^p(\mathbb{R}^n)}$$

for all $u \in L^p(\mathbb{R}^n)$.
Hint: Adapt the ideas of the proof given in class of a similar, albeit simpler, inequality.

Problem 2. Prove that $u \in W_0^{1,p}(\Omega)$ ($1 \leq p < \infty$, $\partial\Omega$ a C^1 boundary, Ω bounded), if and only if $Tu = 0$, where T is the trace operator.

Problem 3. Prove the uniqueness statement in the proof of the "Riesz representation for Sobolev spaces" (the part that was not done in class).

2. Solutions

Solution 1. All inequalities are a direct consequence of the definitions but the last one. Using Hölder's inequality we have

$$\begin{aligned}\int_{\mathbb{R}^n \setminus B_r(x)} |u(y)||x-y|^{k-n}\, dy &\leq \|u\|_{L^p(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n \setminus B_r(x)} |x-y|^{(k-n)p'}\, dy \right)^{\frac{1}{p'}} \\ &\leq C\|u\|_{L^p(\mathbb{R}^n)} \left(\int_r^\infty t^{(k-n)p'+n-1}\, dt \right)^{\frac{1}{p'}} \\ &\leq C_1 r^{k-\frac{n}{p}} \|u\|_{L^p(\mathbb{R}^n)},\end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and we used that $kp < n$.

For $\tau > 0$, let r be such that $C_1 r^{k-\frac{n}{p}} \|u\|_{L^p(\mathbb{R}^n)} = \frac{\tau}{2}$. If

$$G_k * |u|(x) = \int_{\mathbb{R}^n} |u(y)||x-y|^{k-n}\, dy > \tau,$$

then

$$\chi_r G_k * |u|(x) = \int_{B_r(x)} |u(y)||x-y|^{k-n}\, dy > \frac{\tau}{2}. \tag{2.1}$$

Thus,

$$\begin{aligned}
|\{x \mid G_k * |u|(x) > \tau\}| &\leq \left|\{x \mid \chi_r G_k * |u|(x) > \frac{\tau}{2}\}\right| \\
&\leq \left(\frac{2}{\tau}\right)^p \|\chi_r G_k * |u|\|^p_{L^p(\mathbb{R}^n)} \\
&\leq \left(\frac{r^{\frac{n}{p}-k}}{C_1\|u\|_{L^p(\mathbb{R}^n)}}\right)^p C r^{kp}\|u\|^p_{L^p(\mathbb{R}^n)} \\
&= C_2 r^n,
\end{aligned}$$

where we used the similar lemma about convolutions proved in class. Since

$$r^n = \left(\frac{2C_1}{\tau}\|u\|_{L^p(\mathbb{R}^n)}\right)^{p^*}$$

we have that

$$|\{x \mid G_k * |u|(x) > \tau\}| \leq C_2 \left(\frac{2C_1}{\tau}\|u\|_{L^p(\mathbb{R}^n)}\right)^{p^*}.$$

Therefore, the map

$$u \mapsto G_k * |u|$$

is of weak type (p, p^*).

The values of p satisfying the our assumptions form an open set, so we can find p_1 and p_2 in that set and $0 < \theta < 1$ such that

$$\begin{aligned}
\frac{1}{p} &= \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \\
\frac{1}{p^*} &= \frac{1}{p} - \frac{k}{n} = \frac{1-\theta}{p_1^*} + \frac{\theta}{p_2^*}.
\end{aligned}$$

Because $p^* > p$, the Marcinkiewicz interpolation theorem implies that the map $u \mapsto G_k * |u|$ is bounded from $L^p(\mathbb{R}^n)$ to $L^{p^*}(\mathbb{R}^n)$.

Solution 2. Since the trace is continuous from $W^{1,p}(\Omega)$ to $W^{1,p}(\partial\Omega)$ and every element in $C_c^\infty(\Omega)$ has zero trace, we conclude that $Tu = 0$ for $u \in W_0^{1,p}(\Omega)$.

Suppose now that $u \in W^{1,p}(\Omega)$ satisfies $Tu = 0$. As usual, we can reduce the proof to the case $\Omega = \{x^n > 0\}$. Extend u to be zero outside Ω and denote $\tilde{u}$ this extension. Let $u_j \in C^\infty(\overline{\Omega})$ be a sequence converging to u in $W^{1,p}(\Omega)$. The difference

$$\int_{\mathbb{R}^n} \tilde{u}_j D^\alpha \varphi - (-1)\int_{\mathbb{R}^n} \widetilde{D^\alpha u_j}\varphi$$

is a sum of integrals of the form

$$\int_{\mathbb{R}^{n-1}} u_j(x^1, \ldots, x^{n-1}, 0)\varphi(x^1, \ldots, x^{n-1}, 0)$$

which tends to zero by the assumption of zero trace on u. Hence,

$$\int_{\mathbb{R}^n} \tilde{u} D^\alpha \varphi - (-1)\int_{\mathbb{R}^n} \widetilde{D^\alpha u}\varphi = 0$$

and $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$. The result now follows from the following claim, which we prove below: $u \in W_0^{k,p}(\Omega)$ if and only if the zero extension of u belongs to $W^{k,p}(\mathbb{R}^n)$.

It is not difficult to see that if $u \in W_0^{k,p}(\Omega)$ then $\tilde{u} \in W^{k,p}(\mathbb{R}^n)$. To prove the converse, we argue as in the proof of approximation of Sobolev functions by smooth functions up to the boundary, producing the functions u_j. Translate $\tilde{u}_j$ by $\tilde{u}_{j,t}(x) = \tilde{u}_j(x - ty)$, where y is as in that proof (but there we translated by $+$ whereas here we translate by $-$). The translation $x - ty$ moves the support of $\tilde{u}_j$ to inside Ω so $u_{j,t}$ belongs to $W^{k,p}(\mathbb{R}^n)$ since $\tilde{u}_{j,t}$ does. The restriction of $u_{j,t}$ to Ω belongs to $W_0^{k,p}(\Omega)$ since $u_{j,t}$ vanishes outside a compact subset of Ω and these restrictions converge to u_j as $t \to 0^+$.