VANDERBILT UNIVERISTY
MATH 8110 — THEORY OF PARTIAL DIFFERENTIAL EQUATIONS
HW 5

Unless stated otherwise, the notation below is as in class.

1. Problems

Problem 1. Let $u \in W^1(\Omega)$. Show that $Du = 0$ a.e. on any set where u is constant.

Problem 2. Is the converse of the previous question true?

Problem 3. Suppose that all weak derivatives of u order k exist. Show that all weak derivatives of order $k-1$ exist. (In class, we defined $W^k(\Omega)$ as the space of functions whose weak derivatives up to order k exits. This problem shows that we could have defined it as the space whose all weak derivatives of order k exist, and we would get the same result.)

2. Solutions

Solution 1. This follows from $Du = Du^+ - Du^-$.

Solution 2. Yes, u will be constant a.e. on connected sets. Using the regularization, from $Du = 0$ we have $0 = (Du)_\varepsilon = Du_\varepsilon$ so $u_\varepsilon = c_\varepsilon =$ constant depending on ε. Since $u_\varepsilon \to u$ in $L^1_{loc}(\Omega)$, the numerical constants c_ε converge in the limit $\varepsilon \to 0^+$.

Solution 3. The proof is somewhat technical and long and involves ideas from functional analysis and distribution theory. We will only provide a sketch.

Let $\mathscr{W}^{k,p}(\Omega)$ denote the set of distributions on Ω with the property that all distributional derivatives of order k belong to $L^p(\Omega)$. Let $\Omega' \subset \Omega'' \subset \Omega$ be open sets such that $\operatorname{dist}(\partial\Omega', \partial\Omega'') > \varepsilon$ and $\operatorname{dist}(\partial\Omega'', \partial\Omega) > \varepsilon$ for some fixed $\varepsilon > 0$. Let $\psi \in C_c^\infty(\Omega)$ be such that $\psi = 1$ on Ω''. Let $u \in \mathscr{W}^{k,p}(\Omega)$ and set $T := \psi u$. Take $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}(\varphi) \subset B_\varepsilon(0)$ and $\varphi = 1$ in a neighborhood of the origin.

Consider the polyhamormonic operator Δ^k in $\mathbb{R}^n$, whose fundamental solution is

$$\Gamma(x) = \begin{cases} c|x|^{2k-n}, & \text{for } 2k < n \text{ or for odd } n \text{ such that } n \leq 2k, \\ c|x|^{2k-n}\ln|x|, & \text{for even } n \leq 2k, \end{cases}$$

where c is a constant chosen such that $\Delta^k\Gamma = \delta$, where δ is the Dirac-delta. Then

$$\Delta^k(\varphi\Gamma) = \delta + \zeta$$

where $\zeta \in C_c^\infty(\mathbb{R}^n)$. It follows that

$$T + \zeta * T = \sum_{|\alpha|=k} \frac{k!}{\alpha!} D^\alpha(\varphi\Gamma) * D^\alpha T.$$

Observe that $\zeta * T \in C^\infty(\mathbb{R}^n)$. Using the Leibniz rule for multi-indices we find that over Ω''

$$D^\alpha T = \psi D^\alpha u,$$

so that

$$D^\alpha(\varphi\Gamma) * D^\alpha T = D^\alpha(\varphi\Gamma) * (\psi D^\alpha u)$$

over Ω'. From the theory of singular integrals it follows that the corresponding singular integral operator applied to $\psi D^\alpha u$ is continuous in $L^p(\Omega')$. Thus $u \in L^p_{loc}(\Omega)$, i.e., $\mathscr{W}^{k,p}(\Omega) \subset L^p_{loc}(\Omega)$.

Consider now $u \in L^p_{loc}(\Omega)$, so $u \in L^p(\widetilde{\Omega})$ for any open $\widetilde{\Omega} \subset\subset \Omega$, and let $T = D^\beta u$ be its distributional derivative, $|\beta| = k - \ell$, $1 \leq \ell \leq k-1$. By assumption $D^\alpha T \in L^p(\widetilde{\Omega})$, $|\alpha| = \ell$, i.e., $T \in \mathscr{W}^{\ell,p}(\widetilde{\Omega})$. By the foregoing $T = D^\beta u \in L^p_{loc}(\Omega)$.