VANDERBILT UNIVERISTY
MATH 8110 — THEORY OF PARTIAL DIFFERENTIAL EQUATIONS
HW 3

1. Problems

Problem 1. Prove that harmonic functions are analytic.

Problem 2. Prove Liouville's theorem for harmonic functions in $\mathbb{R}^n$.

Problem 3. Prove Harnack's inequality for (non-negative) harmonic functions.

The remaining questions are about the heat equation in n-dimensions, i.e.,

$$u_t - \Delta u = 0 \text{ in } (0,\infty) \times \mathbb{R}^n. \tag{1.1}$$

Problem 4. Look for a solution to (1.1) in the form

$$u(t,x) = t^{-\alpha} v(t^{-\beta} x), \tag{1.2}$$

where α and β will be chosen and v will be determined. More precisely, proceed as follows:

(a) Show that plugging (1.2) into (1.1) produces

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y \cdot \nabla v(y) + t^{-(\alpha+2\beta)} \Delta v(y) = 0, \tag{1.3}$$

where $y := t^{-\beta} x$.

(b) Set $\beta = \frac{1}{2}$ in (1.3) to obtain

$$\Delta v(y) + \frac{1}{2} y \cdot \nabla v(y) + \alpha v(y) = 0. \tag{1.4}$$

(c) Assume that v is radially symmetric, i.e.,

$$v(y) = w(r), \tag{1.5}$$

where w is to be determined. Show that in this case (1.4) becomes

$$w'' + \frac{n-1}{r} w' + \frac{1}{2} r w' + \alpha w = 0. \tag{1.6}$$

(d) Set $\alpha = \frac{n}{2}$ in (1.6) to find

$$(r^{n-1} w')' + \frac{1}{2}(r^n w)' = 0. \tag{1.7}$$

(e) From (1.7), conclude that

$$r^{n-1} w' + \frac{1}{2} r^n w = A, \tag{1.8}$$

where A is a constant.

(f) Set $A = 0$ in (1.8) and conclude that

$$w(r) = B e^{-\frac{1}{4} r^2}, \tag{1.9}$$

where B is a constant.

(g) Combine (1.2), (1.5), (1.9), and take into account the choices of α and β, to conclude that

$$u(t,x) = \frac{B}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}},\ t > 0, \tag{1.10}$$

is a solution to (1.1).

Problem 5. Recall that

$$\Gamma(t,x) := \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, & t > 0, x \in \mathbb{R}^n, \\ 0, & t < 0, x \in \mathbb{R}^n, \end{cases}$$

is called the *fundamental solution of the heat equation.* Note that for $t > 0$, $\Gamma(t,x)$ is simply (1.10) with a specific choice of the constant B. In particular, $\Gamma(t,x)$ is a solution of (1.1).

This choice of B is to guarantee Γ to integrate to 1, i.e., using the fact that

$$\int_{\mathbb{R}^n} e^{-|x|^2}\, dx = \pi^{\frac{n}{2}}, \tag{1.11}$$

show that for each $t > 0$

$$\int_{\mathbb{R}^n} \Gamma(t,x)\, dx = 1.$$

(You do *not* have to show (1.11).)

Problem 6. Consider the initial-value problem for the heat equation:

$$u_t - \Delta u = 0, \qquad \text{in } (0,\infty) \times \mathbb{R}^n, \tag{1.12a}$$
$$u(0,x) = g(x),\ \ x \in \mathbb{R}^n. \tag{1.12b}$$

In (1.12), assume that $g \in C^0(\mathbb{R}^n)$ and that there exists a constant $C > 0$ such that $|g(x)| \le C$ for all $x \in \mathbb{R}^n$.

Recall that we showed existence of a solution by defining

$$u(t,x) := \int_{\mathbb{R}^n} \Gamma(t, x-y) g(y)\, dy,\ t > 0, x \in \mathbb{R}^n. \tag{1.13}$$

Show that (1.13) is well-defined.

Problem 7. Provide the details of the proof given in class that $u \in C^\infty((0,\infty) \times \mathbb{R}^n)$, where u is defined by (1.13).

Hint: Use the following fact, that you do *not* need to prove. Let α be a multiindex and $t > 0$. If

$$\int_{\mathbb{R}^n} D_x^\alpha \Gamma(t, x-y) g(y)\, dy$$

is well-defined, then

$$D^\alpha u(t,x) = \int_{\mathbb{R}^n} D_x^\alpha \Gamma(t, x-y) g(y)\, dy,$$

where we write D_x^α on the RHS to emphasize that the differentiation is with respect to the x variable.

Problem 8. Look up the mean value formula and the maximum principle for solutions to the heat equation.

2. Solutions

Solution 1. See section 2.2.3 of Evan's book.

Solution 2. See section 2.2.3 of Evan's book.

Solution 3. See section 2.2.3 of Evan's book.

Solution 4. These are a sequence of straightforward calculations that are done in the class notes.

Solution 5. Set $z = x/\sqrt{4t}$ and change variables to find

$$\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}}\, dx = \int_{\mathbb{R}^n} e^{-|z|^2} (\sqrt{4t})^n \, dz = \pi^{\frac{n}{2}} (4t)^{\frac{n}{2}}.$$

Solution 6. We have

$$|u(t,x)| \leq \frac{C}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}\, dy.$$

Making the change of variables $z = (y-x)/\sqrt{4t}$ we find

$$\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}\, dy = (4t)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-|z|^2}\, dz < \infty.$$

Solution 7. Let $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)$ be an arbitrary multiindex. Then

$$D_x^\alpha \Gamma(t, x-y) = \frac{p(t,x,y)}{t^M} e^{-\frac{|x-y|^2}{4t}}, \tag{2.1}$$

where M is a non-negative constant and p is a polynomial on its arguments (If (2.1) is not clear, take a few derivatives of $\Gamma(t, x-y)$ and see the pattern that emerges.) Then, using the assumption on g,

$$\begin{aligned} \left| \int_{\mathbb{R}^n} D_x^\alpha \Gamma(t,x-y) g(y)\, dy \right| &\leq C \int_{\mathbb{R}^n} |D_x^\alpha \Gamma(t,x-y)|\, dy \\ &\leq C \int_{\mathbb{R}^n} \frac{|p(t,x,y)|}{t^M} e^{-\frac{|x-y|^2}{4t}}\, dy \\ &= \int_{\mathbb{R}^n} \frac{|q(t,x,z)|}{t^N} e^{-|z|^2}\, dz, \end{aligned}$$

where in the last step we changed variables $z = (y-x)/\sqrt{4t}$, N is a non-negative constant, and q is polynomial on its arguments. We claim that there exists a constant $C > 0$, possibly depending on t, such that

$$\frac{|q(t,x,z)|}{t^N} e^{-|z|^2} \leq C e^{-\frac{1}{2}|z|^2}. \tag{2.2}$$

For, (2.2) is equivalent to

$$\frac{|q(t,x,z)|}{t^N} e^{-\frac{1}{2}|z|^2} \leq C. \tag{2.3}$$

For each fixed x and $t > 0$, the function $\frac{|q(t,x,z)|}{t^N} e^{-\frac{1}{2}|z|^2}$ is a continuous function of z, and because the exponential decays faster than any polynomial, we conclude that $\frac{|q(t,x,z)|}{t^N} e^{-\frac{1}{2}|z|^2}$ is bounded in $\mathbb{R}^n$ as a function of z for each fixed x and $t > 0$, which is (2.3). Since the integral of $e^{-\frac{1}{2}|z|^2}$ is finite, we have shown the result in view of the hint and the fact that α, x, and $t > 0$ are arbitrary.

Solution 8. See sections 2.3.2 and 2.3.3 of Evan's book.