VANDERBILT UNIVERISTY
MATH 8110 — THEORY OF PARTIAL DIFFERENTIAL EQUATIONS
HW 1

Unless stated otherwise, the notation and terminology below is the same used in class. Problems 1–4 are very basic, feel free to skip them if you think they are not edifying.

1. Problems

Problem 1. Verify whether the given function is a solution of the given PDE:

(a) $u(x,y) = y\cos x + \sin y \sin x$, $u_{xx} + u = 0$.

(b) $u(x,y) = \cos x \sin y$, $(u_{xx})^2 + (u_{yy})^2 = 0$.

Problem 2. Determine whether the PDEs below are linear or nonlinear:

(a) $\dfrac{\partial^2 u}{\partial t^2} + e^t \dfrac{\partial u}{\partial x} + u = 0$.

(b) $\partial_x u \partial_y u = 1$.

(c) $\dfrac{\partial^2 z}{\partial t^2} + e^t \dfrac{\partial z}{\partial x} + \cos z = 0$.

(d) $(u_{xx})^2 + (u_{yy})^2 = 1$.

Problem 3. Write each PDE below in the form $P(x, u, Du, \dots, D^k u) = 0$, i.e., identify the function P. State if the PDE is homogeneous or non-homogeneous, linear or non-linear.

(a) $u_{tt} - u_{xx} = f$.

(b) $u_y + u u_x = 0$.

(c) $a^{ijk}\partial^3_{ijk} v + v = 0$,
where i, j, k range from 1 to 3.

(d) $u_{xx} + x^2 y^2 u_{yy} = (x+y)^2$.

(e) $u_{xy} + \cos(u) = \sin(xy)$.

Problem 4. Consider a PDE $P(x, u, Du, \dots, D^k u) = 0$. Show that P is a linear map if and only if it can be written as

$$\sum_{|\alpha| \le k} a_\alpha D^\alpha u = f.$$

Thus, an equivalent definition of a linear PDE is that the map P is linear.

Problem 5. Consider Maxwell's equations:

$$\begin{aligned} \operatorname{div} E &= \frac{\varrho}{\varepsilon_0}, \\ \operatorname{div} B &= 0, \\ \frac{\partial B}{\partial t} + \operatorname{curl} E &= 0, \\ \frac{\partial E}{\partial t} - \frac{1}{\mu_0 \varepsilon_0} \operatorname{curl} B &= -\frac{1}{\varepsilon_0} J. \end{aligned}$$

Assume that ϱ and J vanish. Show that Maxwell's equations then imply that E and B satisfy the wave equation:

$$\frac{\partial^2 E}{\partial t^2} - \frac{1}{\varepsilon_0 \mu_0} \Delta E = 0,$$

and

$$\frac{\partial^2 B}{\partial t^2} - \frac{1}{\varepsilon_0 \mu_0} \Delta B = 0.$$

Interpret your result. Can you guess what the constant $\frac{1}{\varepsilon_0 \mu_0}$ must equal to?

Problem 6. Consider Euler's equations:

$$\begin{aligned} \partial_t \varrho + u^i \partial_i \varrho + \varrho \partial_i u^i &= 0, \\ \varrho(\partial_t u^j + u^i \partial_i u^j) + \nabla^j p &= 0, \end{aligned}$$

where we recall that $p = p(\varrho)$. A fluid is called *incompressible* if $\varrho = constant$, in which case we can set $\varrho = 1$. In this case, the equations describing the fluid motion are

$$\begin{aligned} \partial_t u^j + u^i \partial_i u^j + \nabla^j p &= 0, \\ \partial_i u^i &= 0, \end{aligned}$$

which are called the *incompressible Euler equations.* For an incompressible fluid, however, the pressure is no longer given by $p = p(\varrho)$, since the pressure would then be constant, but experiments show that the pressure can vary even if the density remains (approximately) constant. Show that in the case of the incompressible Euler equations, the pressure is given as a solution to

$$\Delta p = -\partial_j u^i \partial_i u^j.$$

Problem 7. Consider the incompressible Euler equations (see previous question):

$$\begin{aligned} \partial_t u^j + u^i \partial_i u^j + \nabla^j p &= 0, \\ \partial_i u^i &= 0. \end{aligned}$$

The *vorticity* ω of the fluid is defined as

$$\omega := \operatorname{curl} u.$$

The vorticity is an important physical quantity; it measures, as the name suggests, "eddies" in the fluid. It is, therefore, important to know how it changes in time and space (i.e., what the dynamics of the vorticity is). Show that ω satisfies the following PDE:

$$\partial_t \omega + \nabla_u \omega - \nabla_\omega u = 0.$$

Above, the operators ∇_u and ∇_ω are defined as follows. For any vector field X, ∇_X is a short hand notation for $X \cdot \nabla$, i.e.,

$$\nabla_X := X \cdot \nabla,$$

where we recall that $X \cdot \nabla$ has been defined in class as

$$X \cdot \nabla = X^i \partial_i.$$

2. SOLUTIONS

Solution 1. (a) Compute $u_{xx}(x,y) = -y\cos x - \sin x \sin y = -u(x,y)$, thus u is a solution.

(b) Compute $u_{xx}(x,y) = -\cos x \sin y$, $u_{yy}(x,y) = -\cos x \sin y$, thus

$$u_{xx}(x,y))^2 + (u_{yy}(x,y))^2 = 2\cos^2 x \sin^2 y \neq 0,$$

hence u is not a solution.

Solution 2. (a) Linear. (b) Nonlinear. (c) Nonlinear. (d) Nonlinear.

Solution 3. In order to find P, it is useful to identify whether the PDE is linear, homogeneous, the unknown function, etc.

(a) Unknown: u. Independent variables: x, t. Order: second. We have

$$P(p_1, \dots, p_9) = p_9 - p_6 - f(p_1, p_2).$$

The equation is linear and non-homogeneous.

(b) Unknown: u. Independent variables: x, y. Order: first. We have

$$P(p_1, \dots, p_5) = p_5 + p_3 p_4.$$

The equation is non-linear (because of the term uu_x) and homogeneous.

(c) It is instructive to consider a slightly more general case, with i, j, k ranging from 1 to n. Unknown: v. Independent variables: $x^1, \dots, x^n$. Order: third. We have

$$P(x_1, \dots, x_n, p, p_1, \dots, p_n, p_{11}, \dots, p_{nn}, \dots, p_{111}, \dots, p_{nnn}) = a^{ijk} p_{ijk} + p.$$

The equation is linear and homogeneous.

(d) Unknown: u. Independent variables: x, y. Order: second. We have

$$P(p_1, \dots, p_9) = p_6 + p_1^2 p_2^2 p_9 - (p_1 + p_2)^2.$$

The equation is linear and non-homogeneous.

(e) Unknown: u. Independent variables: x, y. Order: second. We have

$$P(p_1, \dots, p_9) = p_7 + \cos p_3 - \sin(p_1 p_2).$$

The equation is non-linear (because of $\cos u$) and non-homogeneous.

Solution 4. Denote by P_H the homogeneous part of P.

Suppose the PDE is linear. Thus,

$$P_H(x, u, Du, \dots, D^k u) = \sum_{m=0}^{k} P_m(x, D^m u), \tag{1}$$

where each F_m is a sum of linear functions on derivatives of u of order m, i.e.,

$$P_m(x, D^m u) = \sum_{\ell=1}^{n^m} P_{m\ell}(x, u^{(\ell)}), \tag{2}$$

where each $u^{(\ell)}$ represents one of the n^m possible derivatives of u of order m. Let u and v be two functions for which $P(x,u,Du,\dots,D^k u)$ and $P(x,v,Dv,\dots,D^k v)$ are well-defined, but are otherwise arbitrary, and let a and b be two arbitrary constants. Then

$$P_m(x, aD^m u + bD^m v) = a\sum_{\ell=1}^{n^m} P_{m\ell}(x,u^{(\ell)}) + b\sum_{\ell=1}^{n^m} P_{m\ell}(x,v^{(\ell)})$$

by the linearity of $P_{k\ell}$. Hence

$$P_H(x, au+bv, aDu+bv, \dots, aD^k u + bD^k v) = aP_H(x,u,Du,\dots,D^k u) + bP_H(x,v,Dv,\dots,D^k v).$$

Writing for simplicity $Pu = P_H(x,u,Du,\dots,D^k u)$, we conclude

$$P(au+bv) = aP_H(x,u,Du,\dots,D^k u) + bP_H(x,v,Dv,\dots,D^k v) = aPu + bPv,$$

as desired.

Reciprocally, suppose that P is a linear operator. Then it can be written on the form

$$\begin{aligned} Pu = {} & a^{i_1 i_2\cdots i_m}\partial^k_{i_1 i_2\cdots i_k} u + a^{i_1 i_2\cdots i_{k-1}}\partial^{k-1}_{i_1 i_2\cdots i_{k_1}} u \\ & + a^{i_1 i_2\cdots i_{k-2}}\partial^{k-1}_{i_1 i_2\cdots i_{k_2}} u + \cdots + a^{i_1 i_2}\partial^2_{i_1 i_2} u + a^i\partial_i u + au. \end{aligned}$$

This implies that P_H has the decomposition (1) with each P_m satisfying (2).

Solution 5. Under the assumptions, the equations become

$$\operatorname{div} E = 0, \tag{3}$$

$$\operatorname{div} B = 0, \tag{4}$$

$$\frac{\partial B}{\partial t} + \operatorname{curl} E = 0, \tag{5}$$

$$\frac{\partial E}{\partial t} - \frac{1}{\mu_0\varepsilon_0}\operatorname{curl} B = 0. \tag{6}$$

Take the curl of (5) and note that $\operatorname{curl}\frac{\partial}{\partial t} = \frac{\partial}{\partial t}\operatorname{curl}$ to get

$$\frac{\partial}{\partial t}\operatorname{curl} B + \operatorname{curl}\operatorname{curl} E = 0.$$

But $\operatorname{curl} B = \mu_0\varepsilon_0\frac{\partial E}{\partial t}$ by (6), thus

$$\mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2} + \operatorname{curl}\operatorname{curl} E = 0.$$

Recalling the following identity from multivariable calculus

$$\operatorname{curl}\operatorname{curl} f = \nabla(\operatorname{div} f) - \Delta f,$$

and using (3), we obtain the wave equation for E. The wave equation for B is similarly obtained.

The interpretation is that the electric and magnetic fields propagate in vacuum as waves. From the discussion about the wave equation in class, we conclude that $\frac{1}{\sqrt{\mu_0\varepsilon_0}}$ is the speed of propagation of the electromagnetic waves, which, from physics, we know to be equal to the speed of light (in vacuum).

Solution 6. Taking the divergence of the momentum equation and using that $\partial_i u^i = 0$, we find

$$\begin{aligned} 0 &= \partial_j(\partial_t u^j + u^i\partial_i u^j + \nabla^j p) \\ &= \partial_t\partial_j u^j + \partial_j u^i \partial_i u^j + u^i\partial_i\partial_j u^j + \partial_i\partial^i p \\ &= \partial_j u^i\partial_i u^j + \partial_i\partial^i p, \end{aligned}$$

where we denoted $\partial^i := \delta^{ij}\partial_j$, with δ being the Kronecker-delta symbol defined as $\delta^{ij} = \delta_{ij} = \delta^i_j = 1$ if $i = j$ and 0 otherwise. Noting that $\partial^i\partial_i = \Delta$, we have the result.

Remark. Note that while Euler's equations in principle require functions that are only once differentiable, the above calculation assumed that the functions are in fact twice continuously differentiable.

Solution 7. Denoting by $|\cdot|$ the norm in $\mathbb{R}^3$, observe the following identity:

$$\frac{1}{2}\nabla^i|u|^2 = \frac{1}{2}\nabla^i(u^\ell u_\ell) = u^\ell\partial^i u_\ell = u^\ell\partial_\ell u^i + (u^\ell\partial^i u_\ell - u^\ell\partial_\ell u^i),$$

where ∂^i is as in the last question. Next, compute

$$\begin{aligned} (u\times\omega)^i &= \epsilon^{ijk}u_i\omega_k = \epsilon^{ijk}u_j\epsilon_k{}^{\ell n}\partial_\ell u_n \\ &= (\delta^{i\ell}\delta^{jn} - \delta^{j\ell}\delta^{in})u_j\partial_\ell u_n \\ &= u^n\partial^i u_n - u^\ell\partial_\ell u^i, \end{aligned}$$

where we used the identity

$$\epsilon^{ijk}\epsilon_{k\ell n} = \epsilon^{kij}\epsilon_{k\ell n} = \delta^i_\ell\delta^j_n - \delta^j_\ell\delta^i_n,$$

which can be verified directly. From the foregoing we conclude that

$$\nabla_u u = \frac{1}{2}\nabla|u|^2 - u\times\omega,$$

which implies

$$\operatorname{curl}\nabla_u u = -\operatorname{curl}(u\times\omega).$$

Let us compute the RHS:

$$\begin{aligned} (\operatorname{curl}(u\times\omega))^i &= \epsilon^{ijk}\partial_j\omega_k = \epsilon^{ijk}\partial_j(\epsilon_k{}^{\ell n}\partial_\ell u_n) \\ &= \epsilon^{ijk}\epsilon_k{}^{\ell n}\partial_j u_\ell\omega_n + \epsilon^{ijk}\epsilon_k{}^{\ell n}u_\ell\partial_j\omega_n \\ &= (\delta^{i\ell}\delta^{jn} - \delta^{j\ell}\delta^{in})\partial_j u_\ell\omega_n + (\delta^{i\ell}\delta^{jn} - \delta^{j\ell}\delta^{in})u_\ell\partial_j\omega_n \\ &= \partial^n u^i\omega_n - \underbrace{\partial_\ell u^\ell}_{=0}\omega^i + u^i\underbrace{\partial_n\omega^n}_{=0} - u^j\partial_j\omega^i \\ &= (\nabla_\omega u)^i - (\nabla_u\omega)^i, \end{aligned}$$

which implies the result.