HOMEWORK 1

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Problem 1. Review multivariable calculus, especially the chain rule in several variables and vector identities/operators.

Problem 2. Verify whether the given function is a solution of the given PDE:

(a) $u(x, y) = y\cos x + \sin y \sin x$, $u_{xx} + u = 0$.

(b) $u(x, y) = \cos x \sin y$, $(u_{xx})^2 + (u_{yy})^2 = 0$.

Problem 3. Determine whether the PDEs below are linear or nonlinear:

(a) $\dfrac{\partial^2 u}{\partial t^2} + e^t \dfrac{\partial u}{\partial x} + u = 0$.

(b) $\partial_x u \partial_y u = 1$.

(c) $\dfrac{\partial^2 z}{\partial t^2} + e^t \dfrac{\partial z}{\partial x} + \cos z = 0$.

(d) $(u_{xx})^2 + (u_{yy})^2 = 1$.

Problem 4. Consider Maxwell's equations:

$$\begin{aligned} \operatorname{div} E &= \frac{\varrho}{\varepsilon_0}, \\ \operatorname{div} B &= 0, \\ \frac{\partial B}{\partial t} + \operatorname{curl} E &= 0, \\ \frac{\partial E}{\partial t} - \frac{1}{\mu_0 \varepsilon_0} \operatorname{curl} B &= -\frac{1}{\varepsilon_0} J. \end{aligned}$$

Assume that ϱ and J vanish. Show that Maxwell's equations then imply that E and B satisfy the wave equation:

$$\frac{\partial^2 E}{\partial t^2} - \frac{1}{\varepsilon_0 \mu_0} \Delta E = 0,$$

and

$$\frac{\partial^2 B}{\partial t^2} - \frac{1}{\varepsilon_0 \mu_0} \Delta B = 0.$$

Interpret your result. Can you guess what the constant $\frac{1}{\varepsilon_0 \mu_0}$ must equal to?

HOMEWORK 2

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Problem 1. Consider Euler's equations:

$$\partial_t \varrho + u^i \partial_i \varrho + \varrho \partial_i u^i = 0,$$
$$\varrho(\partial_t u^j + u^i \partial_i u^j) + \nabla^j p = 0,$$

where we recall that $p = p(\varrho)$. A fluid is called *incompressible* if $\varrho = constant$, in which case we can set $\varrho = 1$. In this case, the equations describing the fluid motion are

$$\partial_t u^j + u^i \partial_i u^j + \nabla^j p = 0,$$
$$\partial_i u^i = 0,$$

which are called the *incompressible Euler equations.* For an incompressible fluid, however, the pressure is no longer given by $p = p(\varrho)$, since the pressure would then be constant, but experiments show that the pressure can vary even if the density remains (approximately) constant. Show that in the case of the incompressible Euler equations, the pressure is given as a solution to

$$\Delta p = -\partial_j u^i \partial_i u^j.$$

Problem 2. Consider the incompressible Euler equations (see previous question):

$$\partial_t u^j + u^i \partial_i u^j + \nabla^j p = 0,$$
$$\partial_i u^i = 0.$$

The *vorticity* ω of the fluid is defined as

$$\omega := \operatorname{curl} u.$$

The vorticity is an important physical quantity; it measures, as the name suggests, "eddies" in the fluid. It is, therefore, important to know how it changes in time and space (i.e., what the dynamics of the vorticity is). Show that ω satisfies the following PDE:

$$\partial_t \omega + \nabla_u \omega - \nabla_\omega u = 0.$$

Above, the operators ∇_u and ∇_ω are defined as follows. For any vector field X, ∇_X is a short hand notation for $X \cdot \nabla$, i.e.,

$$\nabla_X := X \cdot \nabla,$$

where we recall that $X \cdot \nabla$ has been defined in class as

$$X \cdot \nabla = X^i \partial_i.$$

Problem 3. Show that in spherical coordinates the Laplacian reads

$$\Delta = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_{S^2},$$

where

$$\Delta_{S^2} := \partial_\phi^2 + \frac{\cos\phi}{\sin\phi}\partial_\phi + \frac{1}{\sin^2\phi}\partial_\theta^2$$

is the Laplacian on the unit sphere, $r \in [0, \infty)$, $\phi \in [0, \pi]$, and $\theta \in [0, 2\pi)$.

Problem 4. Show that the Φ equation in the separation of variables for the Schrödinger equation can be written as

$$\frac{\sin\phi}{\Phi}\frac{d}{d\phi}\left(\sin\phi\frac{d\Phi}{d\phi}\right) - m^2 = -\lambda\sin^2\phi,$$

where $\lambda = \frac{2\mu}{\hbar^2}a$.

HOMEWORK 3

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

In the next two questions, you will be asked to perform some of the computations skipped in class in our derivation of solutions to the Schödinger equation for an electrostatic potential.

Question 1. Consider the ODE

$$\frac{d}{dx}\left((1-x^2)\frac{d\Phi}{dx}\right)+\left(\lambda-\frac{m^2}{1-x^2}\right)\Phi=0. \tag{1}$$

Show that a solution to (1) is given by

$$\Phi(x)=(1-x^2)^{\frac{|m|}{2}}\frac{d^{|m|}P(x)}{dx^{|m|}}, \tag{2}$$

where P solves

$$(1-x^2)\frac{d^2P}{dx^2}-2x\frac{dP}{dx}+\lambda P=0. \tag{3}$$

Hint: Differentiate (3) a few times and observe the resulting pattern, or use induction, to conclude that if P solves (3) then

$$(1-x^2)\frac{d^{|m|+2}P}{dx^{|m|+2}}-2(|m|+1)x\frac{d^{|m|+1}P}{dx^{|m|+1}}+(\lambda-|m|(|m|+1))\frac{d^{|m|}P}{dx^{|m|}}=0. \tag{4}$$

Next, let $\widetilde{\Phi}$ be defined by

$$\Phi(x)=(1-x^2)^{\frac{|m|}{2}}\widetilde{\Phi}(x), \tag{5}$$

where Φ is a solution to (1). Plugging (5) into (1), conclude that $\widetilde{\Phi}$ satisfies

$$(1-x^2)\frac{d^2\widetilde{\Phi}}{dx^2}-2x\left(|m|+1\right)\frac{d\widetilde{\Phi}}{dx}+\left(\lambda-|m|(|m|+1)\right)\widetilde{\Phi}=0. \tag{6}$$

Compare (6) with (4) to obtain the result.

Question 2. Consider the ODE

$$\frac{1}{\varrho^2}\frac{d}{d\varrho}\left(\varrho^2\frac{dR}{d\varrho}\right)+\left(-\frac{1}{4}-\frac{\ell(\ell+1)}{\varrho^2}+\frac{\gamma}{\varrho}\right)R=0. \tag{7}$$

Show that (7) does not admit a non-trivial solution of the form

$$R(\varrho)=\sum_{k=0}^{\infty}a_k\varrho^k. \tag{8}$$

Hint: Plug (8) into (7) to derive

$$-\ell(\ell+1)a_0\varrho^{-2}+\left((2-\ell(\ell+1))a_1+\gamma a_0\right)\varrho^{-1}$$

$$+\sum_{k=0}^{\infty}\left(\left((k+3)(k+2)-\ell(\ell+1)\right)a_{k+2}+\gamma a_{k+1}-\frac{1}{4}a_k\right)\varrho^k=0. \tag{9}$$

From (9), conclude that $a_k = 0$ for all k.

HOMEWORK 4

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Problem 1. Consider the following initial-boundary value problem for the wave equation in one dimension:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, && \text{in } (0,\infty) \times (0,L), \\ u(t,0) = u(t,L) &= 0, && t \geq 0, \\ u(0,x) &= g(x), && 0 \leq x \leq L, \\ \partial_t u(0,x) &= h(x), && 0 \leq x \leq L, \end{aligned}$$

where g and h are given and satisfy the compatibility conditions $g(0) = g(L) = 0 = h(0) = h(L)$. Use separation of variables to show that

$$u(t,x) = \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi ct}{L} + b_n \sin \frac{n\pi ct}{L} \right) \sin \frac{n\pi x}{L},$$

is a solution to the equation and satisfies the boundary conditions, where a_n and b_n are arbitrary coefficients.

Problem 2. Consider the following initial-boundary value problem for the heat equation in one dimension:

$$\begin{aligned} u_t - u_{xx} &= 0, && \text{in } (0,\infty) \times (0,L), \\ u(t,0) = u(t,L) &= 0, && t \geq 0, \\ u(0,x) &= g(x), && 0 \leq x \leq L, \end{aligned}$$

where g is given and satisfies the compatibility conditions $g(0) = g(L) = 0$. Use separation of variables to show that

$$u(t,x) = \sum_{n=1}^{N} a_n e^{-\frac{n^2\pi^2}{L^2}t} \sin \frac{n\pi x}{L},$$

is a solution to the equation and satisfies the boundary conditions, where the a_n's are arbitrary coefficients. What happens when $t \to \infty$? Interpret your answer physically.

Problem 3. Find the Fourier series of the given functions:

(a) $f(x) = x$, $-1 \leq x \leq 1$.

(b) $f(x) = \sin(5x)$, $-\pi \leq x \leq \pi$.

Problem 4. Consider the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Show that $f \in C^0(\mathbb{R})$, that f is differentiable, but $f \notin C^1(\mathbb{R})$.

Problem 5.

(a) Prove that $C^k(I)$ is a vector space.

(b) Prove that the derivative $\frac{d}{dx}$ is a linear map between $C^k(I)$ and $C^{k-1}(I)$. What happens in the case $k = \infty$?

HOMEWORK 5

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Problem 1. The goal of this problem is to prove the following theorem stated in class: Let $g, h \in C^2([0, L])$ satisfy $g(0) = g(L) = 0 = h(0) = h(L)$ and $g''(0) = g''(L) = 0 = h''(0) = h''(L)$. Then, the formal solution

$$u(t,x) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{L} + b_n \sin \frac{n\pi ct}{L} \right) \sin \frac{n\pi x}{L},$$

where a_n and b_n are given by

$$a_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx,$$
$$b_n = \frac{2}{n\pi c} \int_0^L h(x) \sin \frac{n\pi x}{L}\, dx,$$

is a C^2 solution of the initial-boundary value problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, && \text{in } (0,\infty) \times (0, L), \\ u(t,0) = u(t,L) &= 0, && t \geq 0, \\ u(0,x) &= g(x), && 0 \leq x \leq L, \\ \partial_t u(0,x) &= h(x), && 0 \leq x \leq L, \end{aligned}$$

To prove the theorem, proceed as follows.

(a) Show that g and h can be extended to $2L$-periodic C^2 odd functions on $\mathbb{R}$. Call these extensions $\tilde{g}$ and $\tilde{h}$.

(b) Use D'Alembert's formula to solve the initial value problem for the wave equation on $\mathbb{R}$ with data $\tilde{g}$ and $\tilde{h}$. (In class we derived D'Alembert's formula with $c = 1$; here you need the formula for a general c.)

(c) Consider the Fourier series for $\tilde{g}$ and $\tilde{h}$. Plug these into D'Alembert's formula and using trigonometric identities arrive at the expression given by the formal solution for $x \in [0, L]$. Observe that the boundary conditions are satisfied.

(d) In all the above, make sure that you have the correct assumptions to guarantee the convergence of the Fourier series you employ and whatever other theorem you may need to invoke.

Problem 2. In class we saw that if $u_0 \in C^2(\mathbb{R})$ and $u_1 \in C^1(\mathbb{R})$, then the Cauchy problem for the 1d wave equation with data (u_0, u_1) admits a unique C^2 solution. What can you say if $u_0 \in C^k(\mathbb{R})$ and $u_0 \in C^{k-1}(\mathbb{R})$, $k > 2$?

Problem 3. In class we solved the $1d$ wave equation for $t \geq 0$. Making a change of variables $t \mapsto -t$, show that we can also solve the wave equation for negative times. Conclude that D'Alembert's formula is valid for $-\infty < t < \infty$.

Problem 4. Let $F, G \in C^2(\mathbb{R})$. Show that $F(x + ct) + G(x - ct)$ is a C^2 solution to the $1d$ wave equation.

HOMEWORK 6 - PART I

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Problem 1. Consider the following initial-value problem for the wave equation in one dimension:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \ \text{ in } \ (-\infty,\infty)\times(0,\infty),\\ u(x,0) &= f(x),\\ u_t(x,0) &= g(x), \end{aligned} \tag{1}$$

(a) Solve (1) when $f(x) = x^2$ and $g(x) = 0$.

(b) Assume now that $c = 1$ and

$$f(x) = \begin{cases} 1, & -2 \le x \le 0 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$g(x) = \begin{cases} -1, & -1 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Draw a diagram in the (x,t)-plane indicating the different regions where the solution is influenced by the initial condition f and g and the regions where the solution is identically zero. (You do **not** have to find u.) Is the solution to this problem a classical solution?

Problem 2. Consider the Cauchy problem for the $1d$ wave equation with data (u_0, u_1) and then with data $(\tilde{u}_0, \tilde{u}_1)$. Let u and $\tilde{u}$ be the corresponding solutions. Assume that on $[a,b]$ we have $u_0 = \tilde{u}_0$ and $u_1 = \tilde{u}_1$ Prove that $u = \tilde{u}$ in the domain of dependence with base $[a,b]$.

Problem 3. In the proof of existence of solutions to the initial-boundary value problem for the $1d$ wave equation (HW5, problem 1), drop the hypothesis $g''(0) = g''(L) = 0 = h''(0) = h''(L)$. What can you say about the formal solution in this case? (Will it be an actual solution in any sense? Classical, generalized?)

Problem 4. Write each PDE below in the form $F(x, u, Du, \ldots, D^m u) = 0$, i.e., identify the function F. State if the PDE is homogeneous or non-homogeneous, linear or non-linear.

(a) $u_{tt} - u_{xx} = f$.

(b) $u_y + u u_x = 0$.

(c) $a^{ijk}\partial^3_{ijk} v + v = 0$,
where i, j, k range from 1 to 3.

(d) $u_{xx} + x^2 y^2 u_{yy} = (x+y)^2$.

(e) $u_{xy} + \cos(u) = \sin(xy)$.

Problem 5. Consider a PDE $F(x, u, Du, \dots, D^m u) = 0$. Prove that the PDE is linear (as defined in class in terms of linearity of F with respect to some of its entries) if and only if it can be written as

$$\sum_{|\alpha| \leq k} a_\alpha D^\alpha u = f,$$

as stated in class.

HOMEWORK 6 - PART II

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Problem 1. For each set Ω below: (i) describe Ω in words (e.g., the first quadrant, intersection of the ball of radius one with the third quadrant, etc), drawing a picture when possible; (ii) identify $\partial\Omega$, and $\overline{\Omega}$; (iii) identify the area element of the boundary, i.e., dS; (iv) identify the normal to the boundary. The notation $B_r(z)$ is used for the (open) ball of radius r centered at $z \in \mathbb{R}^n$.

(a)

$$\Omega = \left\{ x \in \mathbb{R}^2 \,\middle|\, |x| < 5 \right\}.$$

(b)

$$\Omega = \left\{ x \in \mathbb{R}^2 \,\middle|\, -2 < x_1 < 2, -1 < x_2 < 1 \right\}.$$

(c)

$$\Omega = \left\{ x \in \mathbb{R}^3 \,\middle|\, -1 < x_1 < 1, -1 < x_2 < 1, -1 < x_3 < 1 \right\}.$$

(d)

$$\Omega = \left\{ x \in \mathbb{R}^2 \,\middle|\, -1 < x_1 < 1, -1 < x_2 < 1 \right\} \bigcap B_1(0).$$

(e)

$$\Omega = \left\{ x \in \mathbb{R}^3 \,\middle|\, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \right\}.$$

(f)

$$\Omega = \left\{ x \in \mathbb{R}^3 \,\middle|\, x_3 > 0 \right\} \bigcap B_r(0).$$

(g)

$$\Omega = \left\{ x \in \mathbb{R}^3 \,\middle|\, \langle x, (1,1,1) \rangle = 0 \right\} \bigcap B_1(0).$$

Problem 2. Recall the integration by parts formula in several dimensions:

$$\int_\Omega f \frac{\partial g}{\partial x_i} = -\int_\Omega \frac{\partial f}{\partial x_i} g + \int_{\partial\Omega} f g \nu_i. \tag{1}$$

Use (1) to prove the following formulas:

(a) Green's first identity:

$$\int_\Omega \nabla f \cdot \nabla g = -\int_\Omega f \Delta g + \int_{\partial\Omega} f \nabla g \cdot \nu.$$

(b) Green's second identity:
$$\int_\Omega (f\Delta g - g\Delta f) = \int_{\partial\Omega} (f\nabla g\cdot\nu - g\nabla f\cdot\nu).$$
(c)
$$\int_\Omega \Delta f = \int_{\partial\Omega} \nabla f\cdot\nu.$$
(d) Divergence theorem:
$$\int_\Omega \operatorname{div} F = \int_{\partial\Omega} F\cdot\nu.$$
In all questions below, let Ω be a domain in $\mathbb{R}^n$, i.e., Ω is an open and connected set contained in $\mathbb{R}^n$. For concreteness you can imagine that Ω is the ball of radius one centered at the origin.

Recall that we say that a function u is k-times continuously differentiable if all derivatives up to order k of u exist and are continuous.

Remember that we defined the spaces
$$C^k(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \,\Big|\, u \text{ is } k\text{-times continuously differentiable } \Big\}.$$

Problem 3. Show that $C^k(\Omega)$ is a vector space.

Problem 4. Show that the Laplacian Δ is a linear map between $C^k(\Omega)$ and $C^{k-2}(\Omega)$, $k \geq 2$.

Problem 5. Recall that
$$C^\infty(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \,\Big|\, u \in C^k(\Omega) \text{ for every } k \Big\}.$$
Show that $C^\infty(\Omega)$ is a vector space and that the Laplacian Δ is a linear map from $C^\infty(\Omega)$ to itself.

Problem 6. Give a reasonable argument for why $C^k(\Omega)$ is an infinite-dimensional vector space. You are not asked to provide a mathematical and rigorous proof. Instead, you should use your knowledge of calculus and linear algebra, as well as the material we learned in class, to construct a sensible explanation, even if only an intuitive one (but if you do know a rigorous proof, that is welcome as well).

Problem 7. In class we said Ω has a C^k boundary if $\partial\Omega$ can be written locally as the graph of a C^k function. Make this definition more precise upon using mathematical quantifiers.

Problem 8. Show that if a_{ij} is symmetric in i and j, then $a^i{}_j = a_j{}^i$ so that we can write simply a^i_j, but that this is not the case otherwise.

Problem 9. Let a be a $n \times n$ matrix with entries $a^i{}_j$, where i the row and j the column. Show that the trace of a is given by $a^i{}_i$. If a is invertible with entries $(a^{-1})^i{}_j$, show that
$$a^i{}_j (a^{-1})^j{}_\ell = \delta^i_\ell.$$
Show that the determinant of a is given by
$$\det(a) = \frac{1}{n!} \epsilon_{i_1 i_2 \cdots i_n} \epsilon^{j_1 j_2 \cdots j_n} a^{i_1}{}_{j_1} a^{i_2}{}_{j_2} \cdots a^{i_n}{}_{j_n}.$$

Above, $\epsilon_{i_1 i_2 \cdots i_n}$ is the n-dimensional totally anti-symmetric symbol, defined as $\epsilon_{i_1 i_2 \cdots i_n} = 1$ if $i_1, i_2, \cdots, i_n$ is an even permutation of $1, 2, \cdots, n$, $\epsilon_{i_1 i_2 \cdots i_n} = -1$ if $i_1, i_2, \cdots, i_n$ is an odd permutation of $1, 2, \cdots, n$, and $\epsilon_{i_1 i_2 \cdots i_n} = 0$ otherwise.
Hint: Show the determinant formula only for $n = 2$ and, perhaps, $n = 3$. The general case is too lengthy for you to spend time on this. You can, however, see the general proofs in textbooks if you are interested.

HOMEWORK 7

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Problem 1. Prove the following fact that we used in the construction of solutions to Poisson's equation: let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous, then

$$\lim_{r \to 0^+} \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} f \, dS = f(x).$$

Hint: Consider the difference $f(x) - \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} f \, dS$ and use $\frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} dS = 1$.

Problem 2. In class, we constructed solutions to Poisson's equation in $\mathbb{R}^n$ for $n \geq 3$. Carry out the construction in the case $n = 2$. You do *not* have to do all the steps. Rather, follow what was done in class and point out what changes in $n = 2$. This boils down to slightly modifying some of the estimates for the fundamental solution.

Problem 3. Let u be a non-trivial harmonic function in $\mathbb{R}^n$. Can u have compact support?
Hint: mean value theorem.

Problem 4. Prove the converse of the mean value theorem. I.e., let $u \in C^2(\Omega)$ be such that

$$u(x) = \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u \, dS$$

for each $B_r(x) \subset \Omega$. Show that $\Delta u = 0$ in Ω.
Hint: Assume that $\Delta u(x) \neq 0$ for some $x \in \Omega$. Use the functions $f(r)$, $f'(r)$ used in the proof of the mean value to derive a contradiction.

Problem 5. Give an interpretation of Poisson's equation in an application (e.g., in physics, biology, etc.).

HOMEWORK 8

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Problem 1. Prove the differentiation of moving regions formula stated in class:

$$\frac{d}{d\tau}\int_{\Omega(\tau)} f\,dx = \int_{\Omega(\tau)} \partial_\tau f\,dx + \int_{\partial\Omega(\tau)} f v\cdot\nu\,dS. \tag{1}$$

(See the class notes for the notation and precise assumptions.) For simplicity, prove (1) in the following particular case. Assume that $n=3$ and that the domains $\Omega(\tau)$ are given by a one-parameter family of one-to-one and onto maps $\varphi=\varphi(\tau,x):\Omega\to\Omega(\tau)=\varphi(\tau,\Omega)$, where $\Omega:=\Omega(0)$ and $\varphi(0,\cdot)=\mathrm{id}_\Omega$, where id_Ω is the identity map on Ω, i.e., $\mathrm{id}_\Omega(x)=x$, $x\in\Omega$.

(a) For each fixed τ, consider the change of variables $x=\varphi(\tau,y)$, so that

$$\int_{\Omega(\tau)} f(\tau,x)\,dx = \int_\Omega f(\tau,\varphi(\tau,y))J(\tau,y)\,dy, \tag{2}$$

where $J(\tau,y)$ is the Jacobian of the transformation $x=\varphi(\tau,y)$ for fixed τ.

(b) Show that there exists a on parameter family of vector fields $u(\tau,\cdot)$ such that

$$\partial_\tau\varphi(\tau,x)=u(\tau,\varphi(\tau,x)).$$

(c) Explain why $u=v$ on $\partial\Omega(\tau)$.

(d) Show that

$$\partial_\tau J(\tau,x)=(\operatorname{div} u)(\tau,\varphi(\tau,x))J(\tau,x).$$

(e) Use (2) and the above to compute $\frac{d}{d\tau}\int_{\Omega(\tau)} f$, and do an integration by parts to obtain the result.

Problem 2. Let u be a solution to the Cauchy problem for the wave equation in $\mathbb{R}^n$. Assume that u_0 and u_1 have their supports in the ball $B_R(0)$ for some $R>0$. Show that $u=0$ in the exterior of the region

$$I:=\{(t,x)\in(0,\infty)\times\mathbb{R}^n \,|\, x\in B_{R+t}(0)\}.$$

I is called a domain of influence for that data on $B_R(0)$ (compare with the $1d$ case).

Problem 3. Let u be a solution to the Cauchy problem for the wave equation and assume that u_0 and u_1 have compact support.

(a) Show that the energy

$$E(t):=\frac{1}{2}\int_{\mathbb{R}^n}\left[(\partial_t u)^2+|\nabla u|^2\right]\,dx$$

is well-defined.

(b) Show that

$$E(t) = E(0),$$

i.e., the energy is conserved.

Problem 4. Let u be a solution to the Cauchy problem for the wave equation in $\mathbb{R}^3$ with compactly supported data (i.e., u_0 and u_1 have compact support).

(a) Show that there exists a constant $C > 0$, depending on u_0 and u_1, such that

$$|u(t,x)| \leq \frac{C}{t}, \tag{3}$$

for $t \geq 1$ and $x \in \mathbb{R}^3$. Thus, for each fixed x, u approaches zero as $t \to \infty$, i.e., solutions decay in time.

Hint: Use the formula for solutions in $n = 3$. Since the data has compact support, it vanishes outside $B_R(0)$ for some $R > 0$. This implies an estimate for the area of the largest region within $B_t(x)$ where the data is non-trivial.

(b) Is the estimate (3) sharp? (I.e., can it be improved to show that solutions decay faster in time than $\frac{1}{t}$?)

(c) Do we still get decay if the data does not have compact support?

HOMEWORK 9 - PART I

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Problem 1. Use Duhamel's principle to show that a solution to the inhomoegeneous wave equation in $1d$ with zero data and source term f is give by

$$u(t,x) = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(t-s,y)\,dyds. \tag{1}$$

To do so, first use D'Alembert's formula to conclude that

$$u_s(t,x) = \frac{1}{2}\int_{x-t+s}^{x+t-s} f(s,y)\,dy.$$

Use the definition of u in terms of u_s and change variables to conclude (1).

Problem 2. Use Duhamel's principle to show that a solution to the inhomoegeneous wave equation in $3d$ with zero data and source term f is give by

$$u(t,x) = \frac{1}{4\pi}\int_{B_t(x)} \frac{f(t-|y-x|,y)}{|y-x|}\,dy. \tag{2}$$

(The integrand in (2) is known as the retarded potential.) To do so, first use Kirchhoff's formula for solutions in $n=3$ to conclude that

$$u_s(t,x) = \frac{t-s}{\mathrm{vol}(\partial B_{t-s}(x))}\int_{\partial B_{t-s}(x)} f(s,y)\,dS(y).$$

Use the definition of u in terms of u_s and change variables to conclude (2).

Problem 3. Show that there exists a constant $C>0$ such that for any solution u to the $3d$ wave equation it holds that

$$|u(t,x)| \le \frac{C}{t}\int_{\mathbb{R}^3}(|D^2u_0(y)| + |Du_0(y)| + |u_0(y)| + |Du_1(y)| + |u_1(y)|)\,dy$$

for $t \ge 1$.

Hint: Use Kirchhoff's formula, note that for any function f we have

$$f(y) = f(y)\frac{y-x}{t}\cdot\frac{y-x}{t}$$

on $\partial B_t(x)$, and use one of Green's identities.

Problem 4. Consider continuous dependence on the data for the wave equation in $3d$, where smallness on the data part is measured with respect to the norm

$$\|f\|_2 := \int_{\mathbb{R}^3}(|D^2f(y)| + |Df(y)| + |f(y)|)\,dy.$$

Give a precise formulation of the continuous dependence on the data and prove your statement.

Hint: Use the estimate of problem 3 as the basis for your statement, and give a similar proof (now you have to also account for $t < 1$).

HOMEWORK 9 - PART II

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

In this assignment, you will guided to construct solutions to the initial-value problem for the heat equation in $\mathbb{R}^n$:

$$u_t - \Delta u = 0 \text{ in } (0,\infty)\times\mathbb{R}^n. \tag{1}$$

Although this assignment has the complexity of a small class project, in that it is longer than a typical HW and stands as a self-contained topic, it will be graded as a regular homework.

Unless stated otherwise, the notation below is as in class.

Problem 1. Look for a solution to (1) in the form

$$u(t,x) = t^{-\alpha} v(t^{-\beta}x), \tag{2}$$

where α and β will be chosen and v will be determined. More precisely, proceed as follows:

(a) Show that plugging (2) into (1) produces

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y\cdot\nabla v(y) + t^{-(\alpha+2\beta)}\Delta v(y) = 0, \tag{3}$$

where $y := t^{-\beta}x$.

(b) Set $\beta = \frac{1}{2}$ in (3) to obtain

$$\Delta v(y) + \frac{1}{2} y\cdot\nabla v(y) + \alpha v(y) = 0. \tag{4}$$

(c) Assume that v is radially symmetric, i.e.,

$$v(y) = w(r), \tag{5}$$

where w is to be determined. Show that in this case (4) becomes

$$w'' + \frac{n-1}{r}w' + \frac{1}{2}rw' + \alpha w = 0. \tag{6}$$

(d) Set $\alpha = \frac{n}{2}$ in (6) to find

$$(r^{n-1}w')' + \frac{1}{2}(r^n w)' = 0. \tag{7}$$

(e) From (7), conclude that

$$r^{n-1}w' + \frac{1}{2}r^n w = A, \tag{8}$$

where A is a constant.

(f) Set $A = 0$ in (8) and conclude that

$$w(r) = Be^{-\frac{1}{4}r^2}, \tag{9}$$

where B is a constant.

(g) Combine (2), (5), (9), and take into account the choices of α and β, to conclude that

$$u(t,x) = \frac{B}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}},\ t > 0, \tag{10}$$

is a solution to (1).

The previous question motivates the following definition. The function

$$\Gamma(t,x) := \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, & t > 0, x \in \mathbb{R}^n, \\ 0, & t < 0, x \in \mathbb{R}^n, \end{cases}$$

is called the *fundamental solution of the heat equation.* Note that for $t > 0$, $\Gamma(t,x)$ is simply (10) with a specific choice of the constant B. This choice of B is to guarantee Γ to integrate to 1 (see the next question). In particular, $\Gamma(t,x)$ is a solution of (1).

Problem 2. Use the fact that

$$\int_{\mathbb{R}^n} e^{-|x|^2}\, dx = \pi^{\frac{n}{2}} \tag{11}$$

to show that for each $t > 0$

$$\int_{\mathbb{R}^n} \Gamma(t,x)\, dx = 1.$$

(You do *not* have to show (11).)

We now consider the initial-value problem for the heat equation:

$$u_t - \Delta u = 0, \qquad \text{in } (0,\infty) \times \mathbb{R}^n, \tag{12a}$$

$$u(0,x) = g(x),\ \ x \in \mathbb{R}^n. \tag{12b}$$

Define

$$u(t,x) := \int_{\mathbb{R}^n} \Gamma(t, x-y) g(y)\, dy,\ t > 0, x \in \mathbb{R}^n. \tag{13}$$

For the next questions, in (12), assume that $g \in C^0(\mathbb{R}^n)$ and that there exists a constant $C > 0$ such that $|g(x)| \leq C$ for all $x \in \mathbb{R}^n$.

Problem 3. Show that (13) is well-defined.

Problem 4. Show that $u \in C^\infty((0,\infty) \times \mathbb{R}^n)$, where u is defined by (13).
Hint: Use the following fact, that you do *not* need to prove. Let α be a multiindex and $t > 0$. If

$$\int_{\mathbb{R}^n} D_x^\alpha \Gamma(t, x-y) g(y)\, dy$$

is well-defined, then

$$D^\alpha u(t,x) = \int_{\mathbb{R}^n} D_x^\alpha \Gamma(t, x-y) g(y)\, dy,$$

where we write D_x^α on the RHS to emphasize that the differentiation is with respect to the x variable.

Problem 5. Show that u given by (13) is a solution to the initial-value problem (12).
Hint: Use the following fact, that you do *not* need to prove. For each $x_0 \in \mathbb{R}^n$,

$$\lim_{(t,x)\to(0,x_0)} u(t,x) = g(x_0).$$

Problem 6. In (12), assume further that g has compact support and that $g \geq 0$. Show that for any $t > 0$ and any $x \in \mathbb{R}^n$, $u(t,x) \neq 0$. Explain why this can be interpreted as saying that, for the heat equation, information propagates at infinite speed. Contrast it with the finite speed of propagation for the wave equation.

HOMEWORK 10

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Problem 1. In the equations below, identify the functions $a(t, x, u)$, $b(t, x, u)$, and $c(t, x, u)$ and write the corresponding characteristic system.

(a) $(1 + t^2)\partial_t u + 3\partial_x u + u^2 = 0$.

(b) $\sin(x)e^t u_t + |u|^3 u_x = 0$.

Problem 2. Solve the problem below using the method of characteristics and give a description of the (projected) characteristics.

$$\begin{aligned} x\partial_t u - t\partial_x u - u &= 0, \\ u(0, x) &= h(x). \end{aligned}$$

Problem 3. Does the transversality condition hold for the problem of question 2? What can you say about uniqueness and how is it related to the solution you found?

Problem 4. Solve the following problems. In each case, sketch the characteristic curves, and indicate the region in the xy-plane where the solution is defined.

(a) $u_x + u_y = u^2$,

for (x, y) in the region $\{y \geq 0\}$, with the condition $u(x, 0) = g(x)$, where g is a given function. Find the solution in the case $g(x) = x^2$.

(b) $u_x + u_y + u = 1$,

with the condition $u = \sin x$ on $y = x + x^2$, $x > 0$.

Problem 5. Solve

$$uu_x - uu_y = u^2 + (x + y)^2,$$

with initial condition $u(x, 0) = 1$.

Hint: after writing the characteristic equations, identify an equation satisfied by $x + y$.

HOMEWORK 11

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Problem 1. In class, in our proof of existence and uniqueness of solutions to the Cauchy problem for first-order quasilinear PDEs in one dimension, we employed a theorem for ODEs about existence, uniqueness, and continuous dependence on the data. Give a precise statement of this theorem.

HOMEWORK 12

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Problem 1. Consider the characteristics $(t, x(t, \alpha))$ for Burgers' equation $\partial_t u + u\partial_x u = 0$. Let v be u written with respect to (t, α) coordinates, i.e., $v(t, \alpha) := u(t, x(t, \alpha))$. Show that v satisfies $\partial_t v = 0$, so v is constant in time. Does this not contradict the fact that solutions to Burgers' equation with non-trivial compactly supported date blow up in finite time?

Problem 2. In this question, you will provide a blow-up proof for Burgers' equation different than the one given in class.

(a) Differentiate the equation and show that the variable $\psi := \partial_x u$ satisfies the equation

$$\partial_t \psi + u\partial_x \psi = -\psi^2. \tag{1}$$

(b) Show that (1) implies that $y(t) := \psi(t, x(t, \alpha))$ satisfies the Riccati equation $\dot{y} = -y^2$ along the characteristics $(t, x(t, \alpha))$.

(c) Use your knowledge of ODE to conclude, from the Riccati equation, blow-up for Burgers.

Problem 3. Define

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} := \sup_{x\in\mathbb{R}} |u(t, x)|.$$

Write Burgers' equation as an ODE along the characteristics (similarly to what you did for ψ in the previous problem) to conclude that

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} = \|u(0, \cdot)\|_{L^\infty(\mathbb{R})} = \|h\|_{L^\infty(\mathbb{R})},$$

i.e., the L^∞ norm is conserved over time.

HOMEWORK 13

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Problem 1. Consider the Cauchy problem for Burgers' equation with data given by

$$h(x) = \begin{cases} 1, & x \leq 0, \\ 1 - x, & 0 < x < 1, \\ 0, & x \geq 1. \end{cases}$$

(a) Show that the solution is given by

$$u(t,x) = \begin{cases} 1, & x \leq t, t < 1, \\ \frac{1-x}{1-t}, & t < x < 1, t < 1, \\ 0, & x \geq 1, t < 1. \end{cases}$$

(b) The denominator of $\frac{1-x}{1-t}$ approaches zero when $t \to 1^-$. Does that mean that $|u(t,x)| \to \infty$ as $t \to 1^-$? Does this not contradict your result from question 3 in HW 12? What exactly is becoming singular when the characteristics intersect at $(1,1)$?

(c) Let $0 < \beta < 1$ and define, for $t \geq 1$

$$\tilde{u}(t,x) = \begin{cases} 1, & x < \beta t + 1 - \beta, \\ 0 & x > \beta t + 1 - \beta. \end{cases}$$

Show that v given by

$$v(t,x) = \begin{cases} u(t,x), & 0 \leq t < 1, \\ \tilde{u}(t,x), & t \geq 1, \end{cases}$$

is a weak solution if and only if $\beta = 1/2$. (This was essentially done in class. Here, you have to work out the calculations in more detail, including the case when Ω might intersect the region $\{t \leq 1\}$.)

Problem 2. Show that the function v in the previous problem verifies the Rankine-Hugoniot conditions if an only if $\beta = 1/2$.

Problem 3. Prove that if u is a weak solution that is C^∞ then it is in fact a classical solution.

Problem 4. Formulate the definition of weak solutions, shocks, and the Rankine-Hugoniot conditions for systems of conservation laws. Give a brief sketch of the proof of the Rankine-Hugoniot theorem for systems (you do not have to do all the proof; it suffices to indicate how to modify the $N = 1$ case done in class. Your answer should be two or three sentences long.)

Problem 5. Show that the $1d$ compressible Euler equations form a system of conservation laws.