MATH 2610, EXAMPLES OF SECTIONS 9.4 AND 9.5

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Question 1. Find the eigenvalues and eigenvectors of the following matrices:
(a)

$$A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}.$$

(b)

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Solutions.
a. Start with the characteristic equation

$$\det \begin{bmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{bmatrix} = -(3-\lambda)(2+\lambda) + 4 = 0,$$

whose solutions are the eigenvalues

$$\lambda_1 = 2,\ \lambda_2 = -1.$$

Let us find the corresponding eigenvectors.
$\underline{\lambda_1 = 2}$:

$$\begin{bmatrix} 3-\lambda_1 & -2 \\ 2 & -2-\lambda_1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix},$$

hence we want to solve

$$\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} v_1 = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We find

$$v_1 = a \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

As we saw in class, we can drop the free variable a and write

$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$\underline{\lambda_2 = -1}$:

$$\begin{bmatrix} 3-\lambda_2 & -2 \\ 2 & -2-\lambda_2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix},$$

hence we want to solve

$$\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} v_2 = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We find
$$v_2 = a \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$
Again, we drop the free variable a, obtaining
$$v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$
Summarizing, we have the following eigenvalues and eigenvectors:
$$\lambda_1 = 2,\ v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ \lambda_2 = -1,\ v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$
b. Start with the characteristic equation
$$\det \begin{bmatrix} 1-\lambda & 1 & 2 \\ 1 & 2-\lambda & 1 \\ 2 & 1 & 1-\lambda \end{bmatrix} = (1-\lambda)\Big((2-\lambda)(1-\lambda)-1\Big) - (1-\lambda-2) + 2\Big(1-2(2-\lambda)\Big) = 0.$$
Rearranging,
$$\begin{aligned}(2-\lambda)(1-\lambda)^2 - 1 + \lambda + 1 + \lambda - 6 + 4\lambda &= (2-\lambda)(1-\lambda)^2 - 6(1-\lambda) \\ &= (1-\lambda)\Big((2-\lambda)(1-\lambda)-6\Big) = 0.\end{aligned}$$
The eigenvalues are now easily found to be
$$\lambda_1 = 4,\ \lambda_2 = -1,\ \lambda_3 = 1.$$
Let us find the corresponding eigenvectors.
$\underline{\lambda_1 = 4}$:
$$\begin{bmatrix} 1-\lambda_1 & 1 & 2 \\ 1 & 2-\lambda_1 & 1 \\ 2 & 1 & 1-\lambda_1 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix},$$
so we need to solve
$$\begin{bmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$
Solving the system and ignoring the free variable as before we obtain
$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$
Repeating the process for $\lambda_2 = -1$, $\lambda_3 = 1$ we find, respectively
$$v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},\ v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$
Summarizing, we have the following eigenvalues with corresponding eigenvectors
$$\lambda_1 = 4,\ v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ \lambda_2 = -1,\ v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},\ \lambda_3 = 1,\ v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$