MATH 2610, EXAMPLES OF SECTIONS 4.6 AND 4.7

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Question 1. Knowing that

$$y_1 = x^2, \text{ and } y_2 = x^3$$

are two linearly independent solutions of

$$x^2 y'' - 4xy' + 6y = 0,$$

find the particular solution y_p of the equation

$$x^2 y'' - 4xy' + 6y = x^3. \tag{1}$$

Question 2. By directly plugging in the formula for y_p, namely,

$$y_p(x) = -y_1(x) \int \frac{y_2(x) f(x)}{W(x)} \, dx + y_2(x) \int \frac{y_2(x) f(x)}{W(x)} \, dx, \tag{2}$$

show that it yields a solution to

$$y''(x) + p(x) y'(x) + q(x) y(x) = f(x). \tag{3}$$

Solutions.

1. It is important to notice that (2) holds when the equation is written as in (3), i.e., with the coefficient of y'' equal to 1. Thus, we divide (1) by x^2, obtaining

$$y'' - \frac{4}{x} y' + \frac{6}{x^2} y = x,$$

in which case $f(x) = x$. We have

$$W = y_1 y_2' - y_2 y_1' = x^2 3x^2 - x^3 2x = x^4$$

(notice that $W \neq 0$ since the equation is not defined for $x = 0$). Then

$$\int \frac{y_2(x) f(x)}{W(x)} \, dx = \int \frac{x^3 x}{x^4} \, dx = x,$$

and

$$\int \frac{y_1(x) f(x)}{W(x)} \, dx = \int \frac{x^2 x}{x^4} \, dx = \ln x.$$

We thus find

$$y_p = x^3 (\ln x - 1).$$

2. We have to plug (2) into (3). Since we shall differentiate y_p, it is useful to remember the Fundamental Theorem of Calculus, which gives

$$\Big(\int \frac{y_2(x) f(x)}{W(x)} \, dx \Big)' = \frac{y_2(x) f(x)}{W(x)},$$

and

$$\Big(\int \frac{y_2(x) f(x)}{W(x)} \, dx \Big)' = \frac{y_2(x) f(x)}{W(x)}.$$

Using these formulas and the product rule we find

$$y_p' = -y_1' \int \frac{y_2 f}{W} - y_1 \frac{y_2 f}{W} + y_2' \int \frac{y_1 f}{W} + y_2 \frac{y_1 f}{W},$$

where we write $\int \frac{y_2 f}{W}$ instead of $\int \frac{y_2(x) f(x)}{W(x)}\, dx$ in order to simplify the notation (analogously for the other integral). Taking another derivative

$$y_p'' = -y_1'' \int \frac{y_2 f}{W} - 2y_1' \frac{y_2 f}{W} - y_1 \left(\frac{y_2 f}{W}\right)' + y_2'' \int \frac{y_1 f}{W} + 2y_2' \frac{y_1 f}{W} + y_2 \left(\frac{y_1 f}{W}\right)'.$$

Using y_p, y_p' and y_p'' into the equation we find

$$\begin{aligned} y_p'' + py_p' + qy_p = &-(y_1'' + py_1' + qy_1) \int \frac{y_2 f}{W} + (y_2'' + py_2' + qy_2) \int \frac{y_2 f}{W} \\ &- \frac{y_2 f}{W}(py_1 + 2y_1') + \frac{y_1 f}{W}(py_2 + 2y_2') - y_1 \left(\frac{y_2 f}{W}\right)' + y_2 \left(\frac{y_1 f}{W}\right)'. \end{aligned}$$

Since by hypothesis y_1 and y_2 are solutions of the homogeneous equation,

$$y_1'' + py_1' + qy_1 = 0,$$

and

$$y_2'' + py_2' + qy_2 = 0,$$

so

$$y_p'' + py_p' + qy_p = -\frac{y_2 f}{W}(py_1 + 2y_1') + \frac{y_1 f}{W}(py_2 + 2y_2') - y_1 \left(\frac{y_2 f}{W}\right)' + y_2 \left(\frac{y_1 f}{W}\right)'$$

By the product rule

$$\left(\frac{y_2 f}{W}\right)' = y_2' \frac{f}{W} + y_2 f' \frac{1}{W} + y_2 f \left(\frac{1}{W}\right)',$$

and

$$\left(\frac{y_1 f}{W}\right)' = y_1' \frac{f}{W} + y_1 f' \frac{1}{W} + y_1 f \left(\frac{1}{W}\right)'.$$

Therefore

$$\begin{aligned} y_p'' + py_p' + qy_p = &- \frac{y_2 f}{W}(py_1 + 2y_1') + \frac{y_1 f}{W}(py_2 + 2y_2') \\ &- y_1 y_2' \frac{f}{W} - y_1 y_2 f' \frac{1}{W} - y_1 y_2 f \left(\frac{1}{W}\right)' \\ &+ y_2 y_1' \frac{f}{W} + y_2 y_1 f' \frac{1}{W} + y_2 y_1 f \left(\frac{1}{W}\right)'. \end{aligned}$$

Notice that the last two terms of the second line cancel with the last two terms of the third line. We are left with

$$\begin{aligned} y_p'' + py_p' + qy_p &= -\frac{y_2 f}{W}(py_1 + 2y_1') + \frac{y_1 f}{W}(py_2 + 2y_2') - y_1 y_2' \frac{f}{W} + y_2 y_1' \frac{f}{W} \\ &= -p\frac{y_2 f}{W} y_1 - 2\frac{y_2 f}{W} y_1' + p\frac{y_1 f}{W} y_2 + 2\frac{y_1 f}{W} y_2' - y_1 y_2' \frac{f}{W} + y_2 y_1' \frac{f}{W}. \end{aligned}$$

The first and third terms on the last line cancel out, and then

$$y_p'' + py_p' + qy_p = -2\frac{y_2 f}{W} y_1' + 2\frac{y_1 f}{W} y_2' - y_1 y_2' \frac{f}{W} + y_2 y_1' \frac{f}{W}$$

$$
\begin{aligned}
&= -2\frac{f}{W}y_2y_1' + 2\frac{f}{W}y_1y_2' - y_1y_2'\frac{f}{W} + y_2y_1'\frac{f}{W} \\
&= \frac{f}{W}y_1y_2' - \frac{f}{W}y_2y_1' = \frac{f}{W}\big(y_1y_2' - y_2y_1'\big) \\
&= f,
\end{aligned}
$$

where in the last step we used that $W = y_1y_2' - y_2y_1'$.