MATH 2610, EXAMPLES OF SECTION 2.3

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Question 1. Find a solution to the initial value problem

$$\begin{cases} (50+t)x' + x - 8t = 400, \\ x(0) = 10, \end{cases}$$

where $t \geq 0$.

Question 2. Consider the two interconnected tanks shown in figure 1. Tank 1 initially contains $30gal$ of water and $25oz$ of salt, while tank 2 initially contains $20gal$ of water and $15oz$ of salt. Water containing $1oz/gal$ of salt flows into tank 1 at a rate of $1.5gal/min$. The mixture flows from tank 1 to tank 2 at a rate of $3gal/min$. Water containing $3oz/gal$ of salt also flows into tank 2 at a rate of $1gal/min$ (from the outside, see picture). The mixture drains from tank 2 at a rate of $4gal/min$, of which some flows back to tank 2 at a rate of $1.5gal/min$, while the remainder leaves the tank.
(a) Let $Q_1(t)$ and $Q_2(t)$, respectively, be the amount of salt in each tank at time t. Write down differential equations and initial conditions that model the flow process. Observe that the system of differential equations is non-homogeneous.
(b) Find the values of $Q_1(t)$ and $Q_2(t)$ for which the system is in equilibrium, i.e., does not change with time.

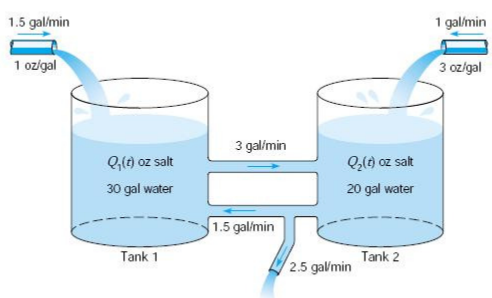

FIGURE 1. Tanks of problem 2.

SOLUTIONS.

Question 1. Since $t \geq 0$, we can divide the equation by $50+t$ as this term is never zero, obtaining

$$\frac{dx}{dt} + \frac{x}{50+t} - \frac{8t}{50+t} = \frac{400}{50+t},$$

or,

$$\frac{dx}{dt} + \frac{x}{50+t} = \frac{400+8t}{50+t} = 8\frac{50+t}{50+t} = 8.$$

The equation

$$\frac{dx}{dt} + \frac{x}{50+t} = 8.$$

is a linear first order equation. As showed in class, the general solution to

$$x' + px = q, \tag{1}$$

is

$$x(t) = \Big(\int q(t) e^{\int p(t)\, dt}\, dt + C \Big) e^{-\int p(t)\, dt}. \tag{2}$$

It is **very important** to notice that (2) can only be applied when the equation is written in the form (1), i.e., with the coefficient multiplying x' being one. That's why we had to first divide the equation by $50 + t$.

In our case, using (2), we find:

$$x(t) = \frac{4(t^2 + 100t + 125)}{50 + t}.$$

Question 2. The volumes of the tanks 1 and 2 are

$$V_1(t) = 30 + 1.5t - 3t + 1.5t = 30,$$
$$V_2(t) = 20 + 3t + 1t - 4t = 20.$$

We can write an equation of the form

$$\text{rate of change of salf in the tank } = \text{ in } - \text{ out },$$

as done in the examples of section 1.1 Then

$$\begin{cases} Q_1' &= 1.5 \times 1 - 3\frac{Q_1}{V_1} + 1.5\frac{Q_2}{V_2}, \\ Q_2' &= 1 \times 3 + \frac{Q_1}{V_1} - 4\frac{Q_2}{V_2}, \end{cases}$$
$$Q_1(0) = 25, Q_2(0) = 15.$$

Or

$$\begin{cases} Q_1' &= 1.5 - \frac{Q_1}{10} + \frac{1.5}{20}Q_2, \\ Q_2' &= 3 + \frac{Q_1}{20} - 4\frac{Q_2}{20}, \end{cases}$$
$$Q_1(0) = 25, Q_2(0) = 15.$$

The equilibrium is given by

$$\begin{cases} 0 &= 1.5 - \frac{Q_1}{10} + \frac{1.5}{20}Q_2, \\ 0 &= 3 + \frac{Q_1}{20} - 4\frac{Q_2}{20}, \end{cases}$$

which gives $Q_1^E = 42$, $Q_2^E = 36$.