MATH 2610, EXAMPLES OF SECTION 2.2

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Question 1. The intensity I of the light at a depth of x meters below the surface of a lake satisfies the differential equations $I' = -1.4I$.
(a) At what depth is the intensity half of the intensity I_0 at the surface of the water?
(b) What is the intensity at a depth of 10 meters?
(c) At what depth will the intensity be 1 % of that at the surface?

Question 2. According to one cosmological theory, there were equal amounts of the two uranium isotopes ${}^{235}U$ and ${}^{238}U$ at the creation of the universe in the big bang. At present there are 137.7 atoms of ${}^{238}U$ for each atom of ${}^{235}U$. Using the half-lives 4.51×10^9 years for ${}^{238}U$ and 7.10×10^8 years for ${}^{235}U$, and assuming a radioactive decay model of the form $x' = kx$, calculate the age of the universe.

SOLUTIONS.

1a. The differential equation is of the form $x' = kx$. This equation is separable, and can be written as

$$\frac{dx}{x} = k\,dt,$$

provided that $x \neq 0$. Integrating

$$\int \frac{dx}{x} = k \int dt \Rightarrow \ln|x| = kt + C,$$

where C is an arbitrary constant of integration. Thus,

$$|x| = e^C e^{kt}.$$

We can remove the absolute value by introducing a sign, i.e., the above tells us that

$$x = \pm e^C e^{kt}.$$

Since C is an arbitrary constant, so is e^C, and therefore we can set $A = \pm e^C$ for some constant A. Our solution then reads

$$x(t) = Ae^{kt}.$$

Next, we notice that the constant A has a very simple interpretation: if x_0 is the value of x at $t = 0$, i.e., $x(0) = x_0$, then

$$x(0) = x_0 = Ae^{k0} \Rightarrow A = x_0,$$

and therefore we can write the solution as

$$x(t) = x_0 e^{kt}.$$

Now we turn our attention to the problem in question. We have that the intensity at a depth of x meters is $I(x) = I_0 e^{-1.4x}$. Then

$$I(x) = \frac{I_0}{2} = I_0 e^{-1.4x} \Rightarrow x = \frac{\ln 2}{1.4} \approx 0.495\, meters.$$

1b. Plugging in, $I(10) = I_0 e^{-1.4\times 10} \approx 8.3 \times 10^{-7}$.

1c. Solving $I_0 e^{-1.4x} = 0.01 I_0$ for x gives $x = \frac{\ln 100}{1.4} \approx 3.29$ meters.

2. Let $N_8(t)$ and $N_5(t)$ be the numbers of ^{238}U and ^{235}U atoms, respectively, t billions of years after the big bang. Since both isotopes follow a radioactive decay model $x' = kx$, whose solution is (see previous problem) $x(t) = x_0 e^{kt}$, we have

$$N_8 = N_0 e^{-kt},$$

and

$$N_5 = N_0 e^{-\ell t},$$

where N_0 is the initial number of atoms of each isotope, which is the same for both ^{238}U and ^{235}U by hypothesis. Notice however that the rates of decay, k and ℓ, differ for these isotopes. Their values are given by

$$N_8(4.51) = \frac{N_0}{2} = N_0 e^{-k\times 4.51} \Rightarrow k = \frac{\ln 2}{4.51},$$
$$N_5(0.71) = \frac{N_0}{2} = N_0 e^{-\ell\times 0.71} \Rightarrow \ell = \frac{\ln 2}{0.71}.$$

We know that for the value of t corresponding to "now" we have $\frac{N_8}{N_5} = 137.7$, hence

$$\frac{N_8}{N_5} = \frac{N_0 e^{-kt}}{N_0 e^{-\ell t}} = e^{(\ell - k)t} = e^{(\frac{\ln 2}{0.71} - \frac{\ln 2}{4.51})t} = 137.7.$$

Solving for t gives

$$t = \frac{\ln 137.7}{\frac{\ln 2}{0.71} - \frac{\ln 2}{4.51}} \approx 5.99.$$

According to this theory, therefore, the universe should be about 6 billion years old.

Note: According to our best current models, the age of the universe is estimated to be about 13.7 billions of years, and the initial ratio of ^{235}U to ^{238}U is estimated to be 1.65 rather than one, as in the exercise[1]. See S. Weinberg, *Cosmology*, Oxford University Press. The interested student can consult the non-technical book *The First Three Minutes: A Modern View Of The Origin Of The Universe*, by the same author.

[1] It makes sense that it is larger than one because three additional neutrons must be added to the progenitor of ^{235}U to from the progenitor of ^{238}U.