MATH 2610, EXAMPLES OF SECTION 12.3

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Question 1. Show that the given system is almost linear near the origin and discuss the type and stability of the critical point at the origin.

$$\begin{cases} x' = -2x + 2xy, \\ y' = x - y + x^2. \end{cases}$$

Solution 1. The system has the form

$$\begin{cases} x' = ax + by + F(x, y), \\ y' = cx + dy + G(x, y), \end{cases}$$

with $a = -2$, $b = 0$, $c = 1$, $d = -1$, $F(x, y) = 2xy$, and $G(x, y) = x^2$. Thus, $ad - bc \neq 0$ (recall that, by definition, an almost linear system must satisfy $ad - bc \neq 0$). We need to check

$$\frac{F(x, y)}{\|(x, y)\|} = \frac{F(x, y)}{\sqrt{x^2 + y^2}} \to 0,$$

and

$$\frac{G(x, y)}{\|(x, y)\|} = \frac{G(x, y)}{\sqrt{x^2 + y^2}} \to 0,$$

as $\|(x, y)\| \to 0$. For $x > 0$ and $y > 0$ we have

$$\begin{aligned} \frac{F(x, y)}{\sqrt{x^2 + y^2}} &= \frac{2xy}{\sqrt{x^2 + y^2}} \\ &= \frac{2xy}{\sqrt{x^2 + y^2}} \\ &= \frac{2}{\sqrt{x^2 + y^2}/\sqrt{x^2 y^2}} \\ &= \frac{2}{\sqrt{\frac{1}{y^2} + \frac{1}{x^2}}} \end{aligned}$$

so that $F(x, y) \to 0$ when $\|(x, y)\| \to 0$, with a similar argument when x, y, or both, are negative. For $G(x, y)$, notice that

$$0 \leq \frac{G(x, y)}{\sqrt{x^2 + y^2}} = \frac{x^2}{\sqrt{x^2 + y^2}} \leq \frac{x^2}{\sqrt{x^2}}.$$

so $G(x, y) \to 0$ when $\|(x, y)\| \to 0$ by the squeeze theorem. Therefore the system is almost linear.

The eigenvalues of the associated linear system are -1 and -2, and we conclude that $(0, 0)$ is an asymptotically stable improper node.