NEGATIVE NORM SOBOLEV SPACES AND APPLICATIONS

MARCELO M. DISCONZI

Abstract. We review the definition of negative Sobolev norms. As applications, we derive a necessary and sufficient condition for existence of weak solutions of linear PDEs, and give Egorov's counter-example of a PDE that is not locally solvable at the origin.

In what follows, the multi-index and sum conventions are adopted. The same letter is used to denote several different constants that appear in the estimates. Some references for the ideas here presented are [1, 2, 3, 4, 5].

1. Negative Sobolev norms.

Let $\Omega \subset \mathbb{R}^n$ be open. Recall that for any $s = 0, 1, \ldots,$ one defines the inner product

$$(u, v)_s = \sum_{|\alpha| \leq s} \int_\Omega \partial^\alpha u \overline{\partial^\alpha v},$$

and the completion of $C^\infty(\Omega)$ w.r.t. the norm $\| \cdot \|_s$ given by the inner product $(\cdot,\cdot)_s$ is the s^{th} Sobolev space $H^s(\Omega)$. Notice that $(\cdot,\cdot)_0$ is the standard L^2-inner product. Although we have introduced the general definition, we will be mostly concerned with real valued functions, in which case the complex conjugation in $\overline{\partial^\alpha v}$ can be dropped.

We first want to characterize the duals of Sobolev spaces[1]. These are among the most important dual spaces for applications in PDE.

For any $v \in L^2(\Omega)$, define $F_v : H^s(\Omega) \to \mathbb{C}$ by $F_v(u) = (u, v)$. It follows that

$$|F_v(u)| \leq \| u \|_0 \| v \|_0 \leq \| u \|_s \| v \|_0,$$

so F_v is a bounded linear functional on $H^s(\Omega)$.

Definition 1.1. Define the negative Sobolev norm $\| \cdot \|_{-s}$ by:

$$\| v \|_{-s} = \| F_v \| = \sup_{u \in H^s(\Omega)} \frac{|(u, v)_0|}{\| u \|_s},$$

where $v \in L^2(\Omega)$. Clearly, $\| \cdot \|_{-s}$ is indeed a norm. Define $H^{-s}(\Omega)$ as the completion of $L^2(\Omega)$ w.r.t. the norm $\| \cdot \|_{-s}$.

Remark 1.2. Since $H^{-s}(\Omega)$ is a completion, it is a Banach space. Later on we shall endow it with an inner product, making it into a Hilbert space.

Remark 1.3. It is important to stress that much of the modern literature uses a slightly different definition, implying that $H^{-s}(\Omega)$ is the dual of $H^s_0(\Omega)$, and not of $H^s(\Omega)$, as in the theorem below. Here, we follow the convention of Lax [4].

Theorem 1.4. *$H^s(\Omega)^* = H^{-s}(\Omega)$.*

[1] Even though $H^s(\Omega)$ is naturally a Hilbert space, for the most part, we will be treating it merely as a Banach space.

Proof. Put $\Lambda = \{F_v \mid v \in L^2(\Omega)\}$, where F_v is as defined above. We claim that Λ is dense in $H^s(\Omega)^*$. Indeed, if this is not the case, take $F \in H^s(\Omega)^* - \overline{\Lambda}$ and then, by (one of the corollaries of) Hahn-Banach, there exists a $\ell \in H^s(\Omega)^{**}$ such that $\ell(F) \neq 0$ and $\ell|_{\overline{\Lambda}} \equiv 0$. By reflexivity, there exists $f_\ell \in H^s(\Omega)$ such that $\ell(\tilde{F}) = \tilde{F}(f_\ell)$ for every $\tilde{F} \in H^s(\Omega)^*$. But then $F_v(f_\ell) = \ell(F_v) = 0$ for all $F_v \in \Lambda$. Thus, $0 = F_v(f_\ell) = (f_\ell, v)$ for all $v \in L^2(\Omega)$, what implies $f_\ell = 0$, a contradiction. Therefore, Λ is dense and $\overline{\Lambda} = H^s(\Omega)^*$.

Now define $\alpha : H^{-s}(\Omega) \to H^s(\Omega)^*$ by $\alpha(v) = \lim_n F_{v_n}$, where $v = \lim_n v_n$, $v_n \in L^2(\Omega)$, $\lim_n v_n$ means limit in $H^{-s}(\Omega)$, and $\lim_n F_{v_n}$ is limit in the operator norm.

First we check that α is well defined. If $v = \lim_n v_n = \lim w_n$, then, since $\| v_n \|_{-s} = \| F_n \|$, we get $\| F_{v_n} - F_{w_n} \| = \| F_{v_n - w_n} \| = \| v_n - w_n \|_{-s} \to 0$ as $n \to \infty$, showing that α is well defined, i.e., $\alpha(v) = \lim_n F_{v_n} = \lim_n F_{w_n}$.

Next, we claim that α is one-to-one and onto. Assume $\alpha(v) = \alpha(w)$. Then $0 = \lim_n \| F_{v_n} - F_{w_n} \| = \lim_n \| v_n - w_n \|_{-s}$ and so $v = w$, showing injectivity. By the density property proved above, α is onto.

Finally, notice that $\| \alpha(v) \| = \| F_v \| = \| v \|_{-s}$, and so α is an isometric isomorphism (i.e., a Banach space isomorphism). □

Using the above theorem, we can extend the notation $(u, v)_0$ to pairs, where $v \in H^{-s}(\Omega)$, and $u \in H^s(\Omega)$, by letting $(u, v)_0$ denote the action of the functional v on u. More explicitly, following the construction of theorem 1.4, one writes $(u, v)_0 = F_v(u)$. Naturally, $F_v(u)$ is the L^2-inner product if $v \in L^2(\Omega)$, but $F_v(u) = \alpha(v)(u)$ if $v \in H^{-s}(\Omega) - L^2(\Omega)$ ($L^2(\Omega) \subset H^{-s}(\Omega)$, see remark below), where α is the map constructed in theorem 1.4, and given by $\alpha(v) = \lim F_{v_n}$, with $v_n \to v$ in $H^{-s}(\Omega)$. Now if $F_{v_n} \to F_v \equiv \alpha(v)$ in the operator norm topology, then $F_{v_n} \equiv (v_n, u)_0 \to F_v(u)$. Indeed

$$F_v(u) = F_v(u) - F_{v_n}(u) + F_{v_n}(u) = F_{v - v_n}(u) + (v_n, u)_0,$$

and $|F_{v-v_n}(u)| \leq \| F_{v-v_n} \| \| u \|_s = \| v - v_n \|_{-s} \| u \|_s \to 0$ when $v_n \to v$ in $H^{-s}(\Omega)$. Thus, when extending the notation $(u, v)_0$, one can also write $(u, v)_0 = \lim (v_n, u)_0$.

Remark 1.5. We have the following facts:

(i) $\| v \|_{-s} \leq \| v \|_0$, so $H^s(\Omega) \subset L^2(\Omega) \subset H^{-s}(\Omega)$.

(ii) Generalized Cauchy-Schwarz inequality:

$$|(u, v)_0| \leq \| u \|_s \| v \|_{-s} \ .$$

(recall that we extended the notation $(u, v)_0$).

Putting all of above together yields:

Theorem 1.6. *Every bounded linear functional on $H^s(\Omega)$ can be represented by F_v, for some $v \in H^{-s}(\Omega)$, i.e., if $F \in H^s(\Omega)^*$ then $F(u) = (u, v)_0$ for all $u \in H^s(\Omega)$.*

Definition 1.7. Let $\beta : H^s(\Omega)^* \to H^s(\Omega)$ be given by the Riesz representation theorem. Then, it is easily checked that the following defines an inner product on $H^{-s}(\Omega)$:

$$(u, v)_{-s} = (\beta \circ \alpha(u), \beta \circ \alpha(v))_s,$$

where α is as in theorem 1.4.

Notice that if $v_n \to v$ in $H^{-s}(\Omega)$, then $F_{v_n} \to F_v$ in the operator norm topology, since $\| F_{v_n} - F_v \| = \| v_n - v \|_{-s} \to 0$; i.e., $F_v(u) = (u, v)_0 = \lim_n (u, v_n)_0$, since $|(u, v - v_n)_0| \leq \| u \|_s \| v - v_n \|_{-s} \to 0$.

Therefore,

$$\sqrt{(v,v)_{-s}} = \sqrt{(\beta\circ\alpha(v),\beta\circ\alpha(v))_s} = \parallel \beta\circ\alpha(v),\beta\circ\alpha(v) \parallel_s$$
$$= \parallel \alpha(v) \parallel = \parallel F_v \parallel = \sup_{u\in H^s(\Omega)} \frac{|(u,v)_0|}{\parallel u \parallel_s} = \parallel v \parallel_{-s},$$

where we used that β is an isometry. Hence $(\cdot,\cdot)_{-s}$ gives (generates) $\parallel\cdot\parallel_{-s}$. Because Hilbert spaces are reflexive, we could conclude from the above that $H^{-s}(\Omega)^* = H^s(\Omega)^{**} = H^s(\Omega)$, but a more specific form of this result will be needed, as stated in the following theorem.

Theorem 1.8. *$H^{-s}(\Omega)^* = H^s(\Omega)$. Moreover, any $G\in H^{-s}(\Omega)^*$ can be represented by a unique $u\in H^s(\Omega)$ via $G(v)=G_u(v)=(u,v)_0$ for all $v\in H^{-s}(\Omega)$.*

Proof. For $u\in H^s(\Omega)$, set $G_u(v)=(u,v)_0$, $v\in H^{-s}(\Omega)$. By the generalized Cauchy-Schwarz inequality, it is seen that that $G_u\in H^{-s}(\Omega)^*$. We claim that $\parallel G_u \parallel = \parallel u \parallel_s$. To see this, first, notice that $|G_u(v)| \leq \parallel u \parallel_s \parallel v \parallel_{-s}$ for every $v\in H^{-s}(\Omega)$, so $\parallel G_u \parallel \leq \parallel u \parallel_s$. For the opposite inequality, consider

$$\parallel G_u \parallel = \sup_{v\in H^{-s}(\Omega)} \frac{|G_u(v)|}{\parallel v \parallel_{-s}} = \sup_{v\in H^{-s}(\Omega)} \frac{|(u,v)_0|}{\parallel v \parallel_{-s}}.$$

By theorem 1.6, $\frac{|(u,v)_0|}{\|v\|_{-s}} = \frac{|F_v(u)|}{\|F_v\|}$, for all $v\in H^{-s}(\Omega)$. Choose v' such that $F_{v'}(u) = \parallel u \parallel_s$ and $\parallel F_{v'} \parallel = 1$; this can be done by one of the corollaries of the Hahn-Banach theorem. Then

$$\parallel G_u \parallel \geq \sup_{F\in H^s(\Omega)^*} \frac{|F(u)|}{\parallel F \parallel} \geq \parallel u \parallel_s .$$

Let $\Lambda = \{G_u \mid u\in H^s(\Omega) \subset H^{-s}(\Omega)^*\}$. An argument similiar to that of theorem 1.4 shows that Λ is dense in $H^{-s}(\Omega)^*$. If a sequence $\{u_n\}$ converges to u in $H^s(\Omega)$ then, for any $v\in H^{-s}(\Omega)$, we have $G_{u_n}(v) = (u_n,v) \to (u,v) = G_u(v)$. Define $\gamma : H^s(\Omega)\to H^{-s}(\Omega)$ by $\gamma(u)=G_u$. Clearly γ is one-to-one and onto, and $\parallel \gamma(u) \parallel = \parallel G_u \parallel = \parallel u \parallel_s$, so γ is an isometric isomorphism. □

2. Applications to PDEs

Definition 2.1. Consider a linear PDE given by $Lu=f$ in Ω, $f\in H^t(\Omega)$, $t\in\mathbb{Z}$. We say that it has a weak solution $u\in H^s(\Omega)$, $s\in\mathbb{Z}$, if $(u,L^*v)_0 = (f,v)_0$ for all $v\in C_c^\infty(\Omega)$, where L^* is the formal adjoint of L.

Remark 2.2. Notice that no boundary conditions are imposed on u since $v\in C_c^\infty(\Omega)$.

Recall that if $Lu = a^{ij}u_{x^ix^j} + b^iu_{x^i} + cu$, which is the case for most applications, then $L^*u = a^{ij}u_{x^ix^j} + (a^{ij}_{x^j} - b^i)u_{x^i} + (c - b^i_{x^i} + a^{ij}_{x^ix^j})u$. The idea is that we can integrate by parts to get L^*. The coefficients are assumed to be sufficiently regular as to justify these calculations and the manipulations below.

Theorem 2.3. *A necessary and sufficient condition for $Lu=f$ to have a weak solution $u\in H^{-s}(\Omega)$, for each $f\in H^{-t}(\Omega)$, is that there exists a constant C such that*

$$\parallel v \parallel_t \leq C \parallel L^*v \parallel_s$$

for all $v\in C_c^\infty(\Omega)$.

Proof. Assume the estimate. Put $X = L^* C_c^\infty(\Omega) \subset H^s(\Omega)$, and consider $F_f \equiv F : X \to \mathbb{R}$ given by $F(L^* v) = (f, v)_0$. Notice that this is well defined because the estimate says that if $\| L^* v \|_s = 0$, then $\| v \|_t = 0$. We claim that F is bounded on the subspace $X \subset H^s(\Omega)$. Estimate

$$|F(L^* v)| \leq |(f, v)_0| \leq \| f \|_{-t} \| v \|_t \leq C \| f \|_{-t} \| L^* v \|_s,$$

which shows boundedness. By Hahn-Banach, F extends to $\tilde{F}$ on all $H^s(\Omega)$, i.e., $\tilde{F} \in H^s(\Omega)^*$. Because $H^s(\Omega)^* = H^{-s}(\Omega)$, by theorem 1.8, there exists a $u \in H^{-s}(\Omega)$ such that $\tilde{F}(w) = (u, w)_0$ for all $w \in H^s(\Omega)$. In particular if $w \in X$,

$$(u, w)_0 = (u, L^* v) = \tilde{F}(L^* v) = F(L^* v) = (f, v)_0,$$

i.e., $(u, L^* v)_0 = (f, v)_0$, for all $v \in C_c^\infty(\Omega)$, showing existence.

Now suppose that for each $f \in H^{-t}(\Omega)$, there exists a $u \in H^{-s}(\Omega)$ such that $Lu = f$ weakly, i.e., $(u, L^* v)_0 = (f, v)_0$ for all $v \in C_c^\infty(\Omega)$. We write $u = u_f$ when we want to emphasize the dependence of u on f. We have

$$|(f, v)_0| = |(u, L^* v)_0| \leq \| u_f \|_{-s} \| L^* v \|_s \leq C_f \| L^* v \|_s,$$

for some constant C_f depending on f. But $v \in H^t(\Omega) \mapsto (f, v)_0 \equiv G_f(v)$ defines a bounded linear functional on $H^t(\Omega)$, i.e., $G_f \in H^t(\Omega)^*$. By Riesz, G_f is represented by an element $\beta(f) \in H^t(\Omega)$, and the map $\beta : H^t(\Omega)^* \to H^t(\Omega)$ is an isometry. Therefore,

$$C_f \| L^* v \|_s \geq |(f, v)_0| = |(\beta(f), v)_t|, \tag{2.1}$$

or

$$\left| (\beta(f), \frac{v}{\| L^* v \|_s})_t \right| \leq C_f.$$

(notice that from the assumption that $Lu = f$ is always solvable, it follows that $L^* v = 0 \Rightarrow v = 0$). Since β is an isomorphism, any $w \in H^t(\Omega)$ is of the form $\beta(f)$ for some f. Thus, we obtain a family of bounded functionals γ_v, given by

$$\gamma_v(w) = (w, \frac{v}{\| L^* v \|_s})_t$$

with the property that, for each v, $|\gamma_v(w)| \leq C_w$. Invoking the principle of uniform boundedness, we conclude that $\| \gamma_v \| \leq C$ for some constant independent of v. Then

$$C \geq \| \gamma_v \| = \sup_{w \in H^t(\Omega)} \frac{|\gamma_v(w)|}{\| w \|_t} \geq \left| (\frac{v}{\| L^* v \|_s}, \frac{z}{\| z \|_t})_t \right|,$$

for any $z \in H^t(\Omega)$. Choosing $z = v$ gives

$$\frac{\| v \|_t^2}{\| L^* v \|_s \| v \|_t} \leq C,$$

or $\| v \|_t \leq C \| L^* v \|_s$, finishing the proof. □

3. A PDE which is not locally solvable at the origin.

As an immediate application of the above theorem we shall give an example of a PDE which has no solution in any neighborhood of the origin. The result is remarkable once we take into account that such PDE is linear and has smooth coefficients. This example is due to Egorov [1, 2].

Definition 3.1. Consider a linear differential operator L. We say that L is locally solvable at the origin if given $f \in C_c^\infty(\Omega_1)$, $\Omega_1 \subset \mathbb{R}^n$ containing the origin, there exists $u \in H^{-s}(\Omega_2)$ solving $Lu = f$ in the weak sense, for some $s \in \mathbb{N}$ and some $\Omega_2 \subset \Omega_1$.

Consider the operator $Lu = u_{tt} - a^2(t)u_{xx} + b(t)u_x$, where $a, b \in C^\infty(\mathbb{R})$ (notice that this is a degenerate hyperbolic equation if a vanishes at some point). The goal is to construct a, b such that the necessary and sufficient condition of theorem 2.3 is violated. We need he following preliminary lemma which refines the necessary condition for this particular L.

Lemma 3.2. *Consider the above operator L. If $Lu = f$ always has a solution in fixed $\Omega \subset \mathbb{R}^2$, $0 \in \Omega$, then there exist a constant C and a $N \in \mathbb{N}$ such that*

$$\| v \|_0 \leq \| L^* v \|_N$$

for all $v \in C_c^\infty(\Omega)$.

Proof. By the theorem 2.3 there exist $s, t \in \mathbb{Z}$ and a constant C such that

$$\| v \|_s \leq C \| L^* v \|_t$$

for all $v \in C_c^\infty(\Omega)$. If $s \geq 0$ then choose $N > t$ and we are done because in this case $\| v \|_0 \leq \| v \|_s$. Otherwise, notice that $\partial_x^\alpha v \in C_c^\infty(\Omega)$ if $v \in C_c^\infty(\Omega)$, so $\| \partial_x^\alpha v \|_s \leq C \| \partial_x^\alpha L^* v \|_t$. Since $L^* v = v_{tt} - a^2(t)v_{xx} + b(t)v_x$ we have

$$\| v_{tt} \|_{s-1} \leq C(\| L^* v \|_{s-1} + \| v_{xx} \|_{s-1} + \| v_x \|_{s-1}) \leq$$
$$C(\| L^* v \|_{s-1} + \| v_x \|_s + \| v \|_s) \leq C \| L^* v \|_{t+1}$$

where we used that we may assume $t \geq s$. Hence

$$\| v \|_{s+1} \leq C(\| v \|_s + \| v_{xx} \|_{s-1} + \| v_{tt} \|_{s-1}) \leq$$
$$C(\| L^* v \|_t + \| L^* v \|_{t+1} \leq C \| L^* v \|_{t+1}$$

We may repeat these arguments to obtain

$$\| v \|_{s+2} \leq C \| L^* v \|_{t+2}, \ldots, \| v \|_{s+n} \leq C \| L^* v \|_{t+n}$$

Then chose n such that $s + n = 0$ and set $N = t + n$ ☐

Theorem 3.3. *Consider again the operator L defined above. There exist $a, b \in C^\infty(\mathbb{R})$ such that given $f \in C^\infty(\Omega)$ with $\Omega \subset \mathbb{R}^2$ some neighborhood of the origin, $Lu = f$ has no solution $u \in H^{-s}(\Omega)$ for any $s \in \mathbb{N}$.*

Remark 3.4. Notice that this does not quite say yet that L is not locally solvable.

Proof. Define

$$a(t) = \begin{cases} e^{-t^2 - \sin^{-2}(\frac{1}{t})} & t > 0 \\ 0 & t \leq 0 \end{cases}$$
$$b(t) = \begin{cases} 2\xi'(t) - a'(t) & t > 0 \\ 0 & t \leq 0 \end{cases}$$

where $\xi = \sin^{-4}(\frac{1}{t})$. We left as an exercise to check that these functions are smooth. Notice that a oscillates very fast with the intervals $I_\mu = (\frac{1}{\pi(\mu+1)}, \frac{1}{\pi\mu})$. The idea is to violate the inequality of lemma 3.2 by constructing a sequence of functions $v_{\mu\lambda} \in C_c^\infty(I_\mu \times (-\frac{2}{\lambda}, \frac{2}{\lambda}))$ which makes he right-hand side of the necessary condition on the above lemma smaller than the left-hand side for large μ, λ.

Recall that $L^* v = v_{tt} - a^2(t)v_{xx} + b(t)v_x$. We look for approximate solutions to $L^* v = 0$ with $v \in C_c^\infty(I_\mu \times (-\frac{2}{\lambda}, \frac{2}{\lambda}))$. It turns out that this can be easily done with separation of variables if we make a simplifying coordinate change to get rid of the v_{xx} term. Namely, set $\bar{t} = t$ and $\bar{x} = x - \int_0^t a(t')dt'$. Clearly this is a smooth change of coordinates in a neighborhood of the origin. Then

$$v_t = \frac{\partial v}{\partial \bar{t}}\frac{d\bar{t}}{dt} + \frac{\partial v}{\partial \bar{x}}\frac{d\bar{x}}{dx} = \frac{\partial v}{\partial \bar{t}} - a\frac{\partial v}{\partial \bar{x}}$$
$$v_{tt} = \frac{\partial^2 v}{\partial \bar{t}^2} - 2a\frac{\partial^2 v}{\partial \bar{x}\partial \bar{t}} + a^2\frac{\partial^2 v}{\partial \bar{x}^2} - a'\frac{\partial v}{\partial \bar{x}}$$
$$v_x = \frac{\partial v}{\partial \bar{x}}\frac{d\bar{x}}{dx} + \frac{\partial v}{\partial \bar{t}}\frac{d\bar{t}}{dx} = \frac{\partial v}{\partial \bar{x}}$$
$$v_{xx} = \frac{\partial^2 v}{\partial \bar{x}^2}$$

So in the new coordinates

$$L^* v = \frac{\partial^2 v}{\partial \bar{t}^2} - 2a\frac{\partial^2 v}{\partial \bar{x}\partial \bar{t}} - (b + a')\frac{\partial v}{\partial \bar{x}}$$
$$= \frac{\partial^2 v}{\partial \bar{t}^2} - 2a\frac{\partial^2 v}{\partial \bar{x}\partial \bar{t}} - 2\xi' a\frac{\partial v}{\partial \bar{x}}$$

From now on we drop the bars on the new coordinates in order to simplify notation. We look for an approximate solution of the form

$$v_{\mu\lambda}(t,x) = \sum_{i=1}^{n} \frac{1}{\lambda^i} z_i(t) w_i(\lambda x)$$

in $C_c^\infty(J_{\mu\lambda})$, where $J_{\mu\lambda} = I_\mu \times (-\frac{1}{\lambda}, \frac{1}{\lambda})$; that is $z_i \in C_c^\infty(I_\mu)$ and $w_i \in C_c^\infty(-1,1)$. To see how to choose z_i and w_i we calculate

$$L^* v_{\mu\lambda} = -\lambda 2a w_0'(z_0' + \xi' z_0) + [z_0'' w_0 - 2a w_1'(z_1' + \xi' z_1)] +$$
$$\frac{1}{\lambda}[z_1'' w_1 - 2a w_2'(z_2' + \xi' z_2)] + \cdots + \frac{1}{\lambda^n} z_n'' w_n$$

It is clear that we should choose w_i so that $w_{i+1}' = w_i$. This can be accomplished by taking $w_n \in C_c^\infty(-1,1)$ and setting

$$w_i = \left(\frac{d}{d\tilde{x}}\right)^{n-i} w_n(\tilde{x}), \text{ where } \tilde{x} = \lambda x$$

Then inductively set

$$z_0(t) = e^{-\xi(t)}$$
$$z_{i+1}(t) = e^{-\xi(t)} \int_0^t \frac{z_i''(s)}{2a(s)} e^{\xi(s)} ds$$

Observe that each $z_i \in C_c^\infty(I_\mu)$ since $\exp(-\sin^{-4}(\frac{1}{t}))$ dominates $\frac{z_i''}{2a}e^{\xi}$ because this last expression is of order $o(t^{-\alpha}\exp(\beta \sin^{-2}(\frac{1}{t}))$ for every $\alpha, \beta > 0$.

With these choices of z_i, w_i we have that $v_{\mu\lambda} \in C_c^\infty(J_{\mu\lambda})$. Moreover

$$\parallel L^* v_{\mu\lambda} \parallel_{H^N(J_{\mu\lambda})} \leq C_{\mu n N} \lambda^{-N}$$

where $C_{\mu n N}$ is independent of λ. By choosing λ sufficiently large we also have

$$\parallel v_{\mu\lambda} \parallel^2_{L^2(J_{\mu\lambda})} \geq \int_{J_{\mu\lambda}} e^{-\xi(t)} w_0(\lambda x) dt dx - D_{\mu n}\lambda^{-1}$$

where $D_{\mu n}$ does not depend on λ nor N. But

$$\int_{\frac{1}{\pi(\mu+1)}}^{\frac{1}{\pi\mu}} e^{-2\sin^{-4}(\frac{1}{t})}dt = \int_{\pi(\mu+1)}^{\pi\mu} e^{-2\sin^{-4}(\tilde{t})}d\tilde{t}$$
$$\geq k^{-4}\int_0^{\pi} e^{-2\sin^{-4}(\tilde{t})}d\tilde{t} \text{ where } \tilde{t} = \frac{1}{t}$$

We also have

$$\int_{-\frac{1}{\lambda}}^{\frac{1}{\lambda}} w_0^2(\lambda x)dx = \int_{-1}^{1} w_0^2(\tilde{x})\lambda^{-1}d\tilde{x} \geq \lambda^{-1}C_n \text{ where } \tilde{x} = \lambda x$$
$$\text{so that } \parallel v_{\mu\lambda}\parallel^2_{L^2(J_{\mu\lambda})} \geq E_{\mu n}\lambda^{-1}$$

It follows that for large λ and μ we can violate the inequality on lemma 3.2 for any given C and N. □

Now we can refine this result in order to obtain that L is not locally solvable at the origin. This is an application of Baire's Category Theorem.

Corollary 3.5. *There exists $f \in C_c(\mathbb{R}^2)$ such that $Lu = f$ has no solution $u \in H^{-s}(\Omega)$ for any $s \in \mathbb{N}$ and any $\Omega \subset \mathbb{R}^2$ containing the origin.*

Proof. For a fixed Ω which contains the origin define

$$X(\Omega) = \{f \in C_c^\infty(\mathbb{R}^2) \mid Lu = f \text{ has a solution } u \in H^{-s}(\Omega) \text{ for some } s \in \mathbb{N}\}$$
$$X_s(\Omega) = \{f \in C_c^\infty(\mathbb{R}^2) \mid Lu = f \text{ has a solution } u \in H^{-s}(\Omega) \text{ such that } \parallel u\parallel_{-s} \leq |s|+1\}$$

(throughout the proof the equality $Lu = f$ is to be understood in the week sense). Notice that $X(\Omega) = \cup_{s=-\infty}^{\infty} X_s(\Omega)$. We are going to show that for each s the space $X_s(\Omega)$ is nowhere dense. For this we first recall the topological properties of $X_s(\Omega)$. The topology on $X_s(\Omega)$ is that given due to the fact that $X_s(\Omega)$ is a Fréchet space and it is generated by open sets

$$\mathcal{U}_{\alpha_1,\ldots,\alpha_k;\epsilon} = \{x \in X_s(\Omega) \mid \parallel x\parallel_{\alpha_1} < \epsilon, \ldots, \parallel x\parallel_{\alpha_k} < \epsilon\}$$

where $\parallel\cdot\parallel_{\alpha_i}$ is the collection of semi-norms. This means that in this topology a sequence $\{x_n\}$ converges to an element x if and only if $\lim_n \parallel x_n - x\parallel_\ell = 0$ for every semi-norm $\parallel\cdot\parallel_\ell$. The collection of semi-norms on $X_s(\Omega)$ is given by the Sobolev norms $\parallel\cdot\parallel_r$, $r \in \mathbb{N}$.

First we claim that $X_s(\Omega)$ is closed. To see this, suppose that $\{f_j\} \subset X_s(\Omega)$ is such that $f_j \to f$. For each j there exists a u_j such that $Lu_j = f_j$ and $\parallel u\parallel_{-s} \leq |s|+1$. Since bounded sequences on Hilbert spaces have weakly convergent sub-sequences, we have that u_{j_k} converges weakly in $H^{-s}(\Omega)$ to some u, so $(u_{j_k}, L^*v) \to (u, L^*v)$ for all $v \in C_c^\infty(\Omega)$; we also have $(f_{j_k}, v) \to (f, v)$ for all $v \in C_c^\infty(\Omega)$. Then $(u, L^*v) = (f, v)$ and so $Lu = f$ weakly. Hence$f \in X_s(\Omega)$ because $\parallel u\parallel_{-s} \leq |s|+1$.

We know show that $X_s(\Omega)$ has empty interior. Let $f \in X_s(\Omega)$ be arbitrary. By the theorem, there exists $\hat{f} \in C_c^\infty(\Omega)$ such that $Lu = \hat{f}$ has no solution for any $s \in \mathbb{N}$. Now, if $f + t\hat{f} \in X_s(\Omega)$ then there exists $w \in H^{-s}(\Omega)$ with $Lw = f + t\hat{f}$. Also, since $f \in X_s(\Omega)$, there exists $z \in H^{-s}(\Omega)$ such that $Lz = f$. But then $tL(w - z) = t\hat{f}$, contradiction. Thus, since $f + t\hat{f} \to 0$ as $t \to 0$ and $f + t\hat{f} \notin X_s(\Omega)$ for any $t \neq 0$ we see that f can not be an interior point.

We conclude from the above that $X(\Omega) = \cup_{s=-\infty}^{\infty} X_s(\Omega)$ is of first category.

Set

$$Y_i = \{f \in C_c^\infty(\mathbb{R}^2) \mid Lu = f \text{ has a solution } u \in H^{-s}(\Omega_i) \text{ for some } s \in \mathbb{N}\}$$

here, $\{\Omega_i\}_{i=1}^{\infty}$ is a countable basis of open sets containing the origing (for example, take all open disks containing the origin with rational radius and rational center). By above each Y_i is of first category. By Baire category theorem $C_c^{\infty}(\mathbb{R}^2) \neq \cup_{i=1}^{\infty} Y_i$. Therefore there exists a nonnull $f \in C_c^{\infty}(\mathbb{R}^2) - \cup_{i=1}^{\infty} Y_i$ and such a f has the desired property. □

4. Techniques for solving boundary value problems.

We now give another application of theorem 2.3. Consider the BVP

$$\begin{cases} Lu = f \\ Bu|_{\partial\Omega} = 0 \end{cases}$$

where L is linear and Bu could be Dirichlet, Neumann, Robin or something different for a particular kind of equation

Definition 4.1. Consider the boundary value problems:

$$\begin{cases} Lu = f \\ Bu|_{\partial\Omega} = 0 \end{cases} (*) \qquad\qquad \begin{cases} L^*v = g \\ B^*v|_{\partial\Omega} = 0 \end{cases} \tag{4.1}$$

and let

$$C_B^{\infty}(\Omega) = \{u \in C^{\infty}(\Omega) \mid Bu|_{\partial\Omega} = 0\}$$
$$C_{B^*}^{\infty}(\Omega) = \{v \in C^{\infty}(\Omega) \mid B^*v|_{\partial\Omega} = 0\}$$

We say that $B^*v|_{\partial\Omega} = 0$ is the adjoint boundary condition for $Bu|_{\partial\Omega} = 0$ if the two following conditions hold: (i) $(u, L^*v) = (f, v)$ for all $v \in C_{B^*}^{\infty}(\Omega)$ implies $u \in C_B^{\infty}(\Omega)$; (ii) $(v, L^*u) = (g, u)$ for all $u \in C_B^{\infty}(\Omega)$ implies $v \in C_{B^*}^{\infty}(\Omega)$

As an example, consider $Lu = \Delta u = f$ with boundary condition $Bu|_{\partial\Omega} = u|_{\partial\Omega} = 0$. We claim that in this case $B^*v|_{\partial\Omega} = v|_{\partial\Omega} = 0$. Indeed, suppose that condition (i) of the above definition holds. Then

$$(f, v) = (u, L^*v) = (Lu, v) + \int_{\partial\Omega} (u\partial_\nu v - v\partial_\nu u) = (f, v) + \int_{\partial\Omega} (u\partial_\nu v - v\partial_\nu u)$$

where ∂_ν is the normal derivative. Since u vanishes on the boundary we get $\int_{\partial\Omega} v\partial_\nu u$ for all $v \in C_{B^*}^{\infty}(\Omega)$, then we must have $B^*v|_{\partial\Omega} = 0$. Analogously we verify condition (ii).

The general idea for solving $(*)$ is the following. Look for an operator M such that we can solve

$$Mu = v, \; v \in C_{B^*}^{\infty}(\Omega)$$

in order to get a $u \in C_B^{\infty}(\Omega)$ (in general, we can have more restrictions on u, such as $Bu|_{\partial\Omega} = 0 +$ extra higher order conditions) such that $(Mu, Lu) \geq C \parallel u \parallel_s^2$. Then

$$\parallel y \parallel_s \parallel L^*v \parallel_{-s} \geq (u, L^*v) = (Lu, v) = (Mu, Lv) \geq \parallel u \parallel_s^2$$

Usually $\parallel v \parallel_{-s} \leq C \parallel u \parallel_s$ as M is of order $\geq s$. We obtain then $\parallel v \parallel_{-s} \leq C \parallel L^*v \parallel_{-s}$.

Now consider the linear functional $F : L^*C_{B^*}^{\infty}(\Omega) \to \mathbb{R}$ given by $F(L^*v) = (f, v)$, f is in $H^s(\Omega)$ or maybe even smooth, depending on the particular problem. Then $F(L^*v) \leq \parallel f \parallel_s \parallel v \parallel_{-s} \leq C \parallel f \parallel_s \parallel L^*v \parallel_{-s}$, so F is bounded on the subspace $L^*C_{B^*}^{\infty}(\Omega) \subset H^{-s}(\Omega)$. By Hahn-Banach it can be extended to $\tilde{F}$ on all $H^{-s}(\Omega)$. By $H^s(\Omega) = H^{-s}(\Omega)^*$ we get that there exists a $u \in H^s(\Omega)$ such that $\tilde{F}(w) = (u, w)$ for all $w \in H^{-s}(\Omega)$. Restricting w bach to $L^*C_{B^*}^{\infty}(\Omega)$ we have

$$(u, L^*v) = \tilde{F}(L^*v) = F(L^*v) = (f, v)$$

for all $C_{B^*}^{\infty}(\Omega)$. By definition of $C_B^{\infty}(\Omega)$, assuming s large enough we can integrate by parts to force u to solve $Lu = f$ and also to satisfy $Bu|_{\partial\Omega} = 0$.

We want to do some examples now. But before that we need one more definition.

Definition 4.2. Define the double (s,r)-Sobolev norm as

$$\| u \|_{(s,r)} = \sum_{\substack{0\le\alpha\le s,\ 0\le\beta\le r\\ \alpha+\beta\le\max(s,r)}} \int_\Omega (\partial_x^\alpha \partial_y^\beta u)^2$$

Then the (s,r)-Sobolev space$H^{(s,r)}(\Omega)$ is the completion of $C^\infty(\Omega)$ w.r.t. the norm $\| \cdot \|_{(s,r)}$.

The above results concerning negative Sobolev norms, including the necessary and sufficient condition for existence, generalize to the double Sobolev norms; see [3].

4.1. **Elliptic equations.** Consider $Lu = \Delta u - u = f$ in $\Omega\{(x,y) \in \mathbb{R}^2 \mid |x| < 1, |y| < 1\}$. We shall prove existence of regular solutions for the Dirichlet and Neumann problems. To determine the correct BVP notice that

$$(u, L^*v) - (Lu, v) = \int_{\partial\Omega} (u\partial_\Omega v - v\partial_\Omega u)$$

so that we have $Bu = u \Leftrightarrow B^*v = v$ for Dirichlet BC and $Bu = \partial_\nu u \Leftrightarrow B^*v = \partial_\nu v$ for Neumann BC. Let

$$Mu = \sum_{k=0}^{s} (-1)^k \partial_x^{2k} u$$

and solve $Mu = v$, $v \in C^\infty_{B^*}(\Omega)$, with

$$\partial_x^{2k} u|_{\partial\Omega_v^\pm} = 0,\ k = 0, 1, \dots, s-1 \text{ for Dirichlet}$$
$$\partial_x^{2k+1} u|_{\partial\Omega_v^\pm} = 0,\ k = 0, 1, \dots, s-1 \text{ for Neumann}$$

and

$$u|_{\partial\Omega_h^\pm} = 0,\ \text{for Dirichlet}$$
$$u_y|_{\partial\Omega_h^\pm} = 0,\ \text{for Neumann}$$

where $\partial\Omega_h^-$ $(\partial\Omega_h^-)$ denotes the bottom (top) horizontal boundary and where $\partial\Omega_v^-$ $(\partial\Omega_v^-)$ denotes the left (right) vertical boundary. Notice that this is possible since for an ODE of order $2s$ we can prescribe $2s$ boundary conditions. Also on $\partial\Omega_h^\pm$ $v = 0$ or $v_y = 0$, so we get $u = 0$ or $u_y = 0$, respectively. Furthermore, notice that $(Mu, Lu) \ge C \| u \|^2_{(s+1,1)}$ for all u having the above boundary conditions. To see that this is indeed the case, it is illustrative to look at the following example

$$\begin{aligned}(-u_{xxxx} + u_{xx} - u, u_{xx} + u_{yy} - u) &= \int_\Omega (u_{xxx}^2 + u_{xx}^2 + u_x^2 + u^2 + u_{xxy}^2 + u_{xy}^2 + u_y^2) \\ + \int_{\partial\Omega} (uu_{xxx}\nu_1 - u_x u_{xx}\nu_1 - uu_x\nu_1 &- u_{xx}u_{xxx} - uu_x\nu_1 - u_{yy}u_{xxx}\nu_1 + u_{xyy}u_{xx}\nu_1) \\ + \int_{\partial\Omega} (-u_{xxy}u_{xx}\nu_2 + u_{yy}u_x\nu_1 &- u_{xy}\nu_2 - u_y u\nu_2) \ge \| u \|^2_{(3,1)}\end{aligned}$$

since all boundary terms vanish (ν_j is the normal in the j-direction).

Now observe that if we set

$$\| v \|_{(-s,-r)} = \sup_{w \in H^{(s,r)}(\Omega)} \frac{|(w,v)|}{\| w \|_{(s,r)}}$$

and $H^{(-s,-r)}(\Omega)$ =completion of $L^2(\Omega)$ w.r.t. the norm $\| \cdot \|_{(-s,-r)}$ in the usual way, then

$$\| v \|_{(-s,0)} \leq \sup_{w \in H^{(s,0)}(\Omega)} \frac{|(w,u)_{(s,0)}|}{\| w \|_{(s,0)}} \leq \| u \|_{(s,0)}$$

As an illustration of how to get this inequality, consider

$$\begin{aligned} -(w,v) &= (w, -u_{xxxx} + u_{xx} - u) = (w,u)_{(2,0)} + \int_{\partial\Omega} (w u_{xxx}\nu_1 - w_x u_{xx}\nu_1 - w u_x \nu_1) \\ &= (w,u)_{(2,0)} \end{aligned}$$

since we can assume that $w|_{\partial\Omega^{\pm}} = 0$, i.e.,

$$\begin{aligned} H^{(3,1)}(\Omega) = & \text{ completion of } C^{\infty}(\Omega) \text{ in the norm } \| \cdot \|_{(3,1)} = \\ = & \text{ completion of } C^{\infty}(\Omega) \cap \{ w \mid w|_{\partial\Omega_v^{\pm}} = 0 \} \text{ in the norm } \| \cdot \|_{(3,1)} \end{aligned}$$

Therefore $|(w,v)| \leq \| w \|_{(2,0)} \| u \|_{(2,0)}$, which is the desired inequality in this particular case.

Continuing the computations, now we have

$$\| u \|_{(s+1,1)} \| L^* v \|_{(-s-1,-1)} \geq (u, L^* v) = (Lu, v) = (Mu, Lu) \geq \| u \|^2_{(s+1,1)}$$

for all $v \in C^{\infty}_{B^*}(\Omega)$. Hence

$$\| v \|_{(-s,0)} \leq \| u \|_{(s+1,1)} \leq \| L^* v \|_{(-s-1,-1)}$$

for all $v \in C^{\infty}_{B^*}(\Omega)$. Summarizing the above, we have the following result

Theorem 4.3. *For each $f \in H^{(s,0)}(\Omega)$ there exists a weak solution $u \in H^{(s+1,1)}(\Omega)$ of $Lu = f$, i.e, $(u, L^* v) = (f, v)$ for all $v \in C^{\infty}_{B^*}(\Omega)$.*

In order to obtain higher regularity in the y-direction we need the following lemma concerning difference quotients, which are defined as

$$u^h(x,y) = \frac{u(x, y+h) - u(x,y)}{h}$$

Lemma 4.4. *(i) If $\Omega' \subset\subset \Omega$ and $u \in H^{(0,1)}(\Omega)$ then*

$$\| u^h \|_{L^2(\Omega')} \leq \| u_y \|_{L^2(\Omega)}$$

(ii) If $u \in L^2(\Omega')$ and $\| u^h \|_{L^2(\Omega')} \leq C$ for all $0 < |h| < \mathrm{dist}(\Omega', \partial\Omega)$ then $u \in H^{(0,1)}(\Omega')$ with $\| u_y \|_{L^2(\Omega')} \leq C$.

Proof. Assume first that u is smooth. Then for $0 < |h| < \frac{1}{2}\mathrm{dist}(\Omega', \partial\Omega)$ we have

$$u(x, y+h) - u(x,y) = h \int_0^1 u_y(x, y+th) dt$$

So

$$|u(x, y+h) - u(x,y)| \leq h \int_0^1 |u_y(x, y+th)| dt$$

Hence, by Jensen's inequality

$$\int_{\Omega'} |u^h|^2 \leq \int_{\Omega'} (\int_0^1 |u_y(x, y+th)|dt)^2 dxdy$$
$$\leq \int_0^1 (\int_{\Omega'} |u_y(x, y+th)|^2 dxdy)dt$$
$$\leq \int_{\Omega'} |u_y(x, y+t_0h)|^2 dxdy \text{ for } t_0 \text{ where the integrand achieves its maximum}$$
$$\leq u_y \|_{L^2(\Omega)}$$

Now, by approximation this holds for $u \in H^{(0,1)}(\Omega)$.

Assume now the hypothesis of (ii). For all $\phi \in C_c^\infty(\Omega')$ we have

$$\int_{\Omega'} u(x, y) \frac{\phi(x, y+h) - \phi(x, y)}{h} dxdy = - \int_{\Omega'} \phi(x, y) \frac{u(x, y) - u(x, y-h)}{h} dxdy$$

where we used integration by parts for difference quotients and a change of variables $y' = y + h$. In other words, we have

$$\int_{\Omega'} u\phi^h = - \int_{\Omega'} u^{-h}\phi$$

By assumption $\sup_h \| u^{-h} \|_{L^2(\Omega')} \leq C$. Using the weakly compactness of bounded sets in Hilbert spaces we get a h_k converging to zero and a $w \in L^2(\Omega')$ such that u^{-h_k} converges in $L^2(\Omega')$ to w. But then

$$\int_{\Omega'} u\phi_y = \int_{\Omega} u\phi_y = \lim_{h_k \to 0} \int_{\Omega} u\phi^{h_k}$$
$$= - \lim_{h_k \to 0} \int_{\Omega'} u^{-h_k}\phi = - \int_{\Omega'} \phi = - \int_{\Omega} w\phi$$

□

We can now obtain full regularity for any solution in the previous theorem.

Corollary 4.5. *For each $f \in H^s(\Omega)$ there exists a unique solution $u \in H^{s+1}(\Omega)$ of $Lu = f$.*

Proof. We know that $u \in H^{(s+1,1)}(\Omega)$ and for all $v \in C^\infty_{B^*}(\Omega)$ we have

$$-(u_y, v_y) = (f - u_{xx} + u, v) = (\bar{f}, v)$$

where $\bar{f} = f - u_{xx} + u \in H^(0, 1)(\Omega)$. Then $-(u_y^h, v_y) = (\bar{f}^h, v)$, and if we choose smooth functions $\{v_i\} \subset C^\infty_{B^*}$ such that $v_i \to u^h$ in $H^{(0,1)}(\Omega)$ we get

$$\| u_y^h \|^2 \leq |(\bar{f}^h, u^h)| \leq \| \bar{f}^h \| \| u^h \| \leq \| \bar{f_y}^h \| \| u^h \| \leq C$$

by (i) of the lemma; and by (ii) we have $u_y \in H^{(0,1)}(\Omega)$, i.e., $u_{yy} \in L^2(\Omega)$. The following then holds in $L^2(\Omega)$

$$u_{yy} = f - u_{xx} - u \in H^{(s-2,1)}(\Omega)$$

We can then differentiate w.r.t. y to get

$$u_{yyy} = (f - u_{xx} - u)_y \in H^{(s-3,1)}(\Omega)$$

and continue to boot strap to get $u \in H^{s+1}(\Omega)$.

Uniqueness follows since now $u \in C^{s-1}(\Omega)$ and $(-u, Lu) \geq \| u \|_1^2$ holds as $v \in C^\infty_{B^*}(\Omega)$ implies that $u|_{\partial\Omega} = 0$ or $\partial_\nu u|_{\partial\Omega} = 0$. □

Remark 4.6. One draw back of this method is that it does not give the best result, i.e., the best is $f \in C^{k,\alpha}(\Omega)$ implying $u \in C^{k+2,\alpha}(\Omega)$, or $f \in H^s(\Omega)$ implying $u \in H^{s+2}(\Omega)$.

4.2. **Hyperbolic equations.** Here we consider the Cauchy problem for the wave equation

$$Lu = u_{tt} - u_{xx} + u = f$$
$$u(x,0) = u_t(x,0) = 0,\ u(\pm 1, t) = 0 \text{ or } u_x(\pm 1, t) = 0$$

and its adjoint boundary value problem

$$L^* v = v_{tt} - v_{xx} + v = g$$
$$v(x,1) = v_t(x,1) = 0,\ v(\pm 1, t) = 0 \text{ or } v_x(\pm 1, t) = 0$$

The wave equation represents the motion of a string (or waves) with fixed ends (for the Dirichlet boundary conditions) under the influence of an external force f. If we tried the full Dirichlet or Neumann problem as for the Laplace equation we would not be able to prove existence using this method.

Let

$$Mu = a(t) \sum_{k=0}^{s} (-1)^k \partial_x^{2k} u_t$$

where $a(t) = -e{-}t$ (we really only need $a'(t) > 0$, $a(t) < 0$) and solve

$$Mu = v,\ v \in C^\infty_{B^*}(\Omega) \text{ with } u(x,0) = u_t(x,0) = 0$$
$$\partial_x^{2k} u(\pm 1, t) = 0,\ k = 0, 1, \ldots, s-1 \text{ fot Dirichlet}$$
$$\partial_x^{2k+1} u(\pm 1, t) = 0,\ k = 0, 1, \ldots, s-1 \text{ fot Neumann}$$

Notice that this can be done using standard theory for ODEs. Then observe that

$$(Mu, Lu) \geq \| u \|^2_{(s+1,1)}$$

for all $u \in C^\infty_B(\Omega) = \{u$ having the above boundary conditions $\}$. For example:

$$\begin{gathered}
(\ u_{tt} - u_{xx} + u\ ,\ a(t)(-u_{txxxx} + u_{txx} - u_t)\ ) = \\
\int_\Omega \frac{a'(t)}{2} \Big(u_{xxx}^2 + u_{xxt}^2 + u_{xt}^2 + 2u_{xx}^2 + u_t^2 + 2u_x^2 + u^2 \Big) + \\
\int_{\partial\Omega} \Big(a u_x u_t \nu_1 - \frac{a}{2} u_t^2 \nu_2 - \frac{a}{2} u_x^2 \nu_2 - \frac{a}{2} u^2 \nu_2 - \frac{a}{2} u_{xx}^2 \nu_2 + a u_{tt} u_{xt} \nu_1 - \frac{a}{2} u_{xt}^2 \nu_2 + \\
a u u_{xt} \nu_1 - \frac{a}{2} u_x^2 \nu_2 - a u_{tt} u_{txx} \nu_1 + a u_{xtt} u_{txx} \nu_1 - \frac{a}{2} u_{xxt}^2 \nu_2 + \\
a u_{xx} u_{txxx} \nu_1 - a u_{xxx}^2 \nu_2 - a u u_{xxxt} \nu_1 + a u_x u_{xxt} \nu_1 - a u_{xx}^2 \nu_2 \Big) \\
\geq C \| u \|^2_{(3,1)}
\end{gathered}$$

since all boundary terms vanish or are non-negative for $u \in C^\infty_B(\Omega)$. Furthermore, as in the elliptic case

$$\| v \|_{(-s,0)} \leq C \sup_{w \in H^{(s,0)}(\Omega)} \frac{|(w, u_t)|}{\| w \|_{(s,0)}} \leq C \| u \|_{(s,1)}$$

For example

$$-(w,v) = -(\ w\ ,\ a(t)(-u_{txxxx} + u_{txx} - u_t)\ ) = (w, a(t)u_t)_{(2,0)}$$
$$+ \int_{\partial\Omega} (awu_{txxx}\nu_1 - aw_x u_{txx}\nu_1 - awu_{tx}\nu_1) = (w, a(t)u_t)_{(2,0)}$$

since we can assume that $w(\pm 1, t) = 0$ as before, so $|(w,v)| \leq C \parallel w \parallel_{(2,0)} \parallel u \parallel_{(2,1)}$ which gives the desired result.

We now have

$$\parallel u \parallel_{(s+1,1)} \parallel L^*v \parallel_{(-s-1,-1)} \geq (u, L^*v) = (Lu, v) = (Mu, Lu) \geq C \parallel u \parallel^2_{(s+1,1)} \text{ for all } v \in C^\infty_{B^*}(\Omega)$$

$((u, L^*v) = (Lu, v)$ is justified because all boundary terms vanish). Hence

$$\parallel v \parallel_{(-s,0)} \leq C \parallel u \parallel_{(s+1,1)} \leq C \parallel L^*v \parallel_{(-s-1,-1)}$$

This proves

Theorem 4.7. *For each $f \in H^{(s,0)}(\Omega)$ there exists a weak solution $u \in H^{(s+1,1)}(\Omega)$ of $Lu = f$, i.e., $(u, L^*v) = (f, v)$ for all $v \in C^\infty_{B^*}(\Omega)$.*

Remark 4.8. Higher regularity may be proven in the same way as for the elliptic case. Then u solves $Lu = f$ pointwise and because $v \in C^\infty_{B^*}(\Omega)$ we have that $u \in C^\infty_B(\Omega)$ has the correct boundary values. Again this method does not necessarily gives the best result.

Exercise: Prove the same result for the parabolic heat equation

$$Lu = u_t - u_{xx} + u = f$$
$$u(x,0),\ u(\pm 1, t) = 0 \text{ or } u_x(\pm 1, t) = 0$$

with adjoint boundary value problem

$$L^*v = v_t - v_{xx} + v = g$$
$$v(x,1),\ v(\pm 1, t) = 0 \text{ or } v_x(\pm 1, t) = 0$$

References

[1] Egorov, Y. V. *Sur un example d'èquation linéaire hyperbolique n'ayant pas de solution.* Journées "Équations aux Dérivées Partielles" (Saint-Jean-de-Monts, 1992), Exp. No. XIV, 6 pp., École Polytech., Palaiseau, 1992.

[2] Egorov, Y. V. *On an example of a linear hyperbolic equation without solutions.* C. R. Acad. Sci. Paris Sr. I Math. 317 (1993), no. 12, 1149-1153.

[3] Khuri, M. A. *Boundary value problems for mixed type equations and applications.* Nonlinear Anal. 74 (2011), no. 17, 6405-6415.

[4] Lax, P. D. *On Cauchys problem for hyperbolic equations and the differentiability of solutions of elliptic equations.* Comm. Pure Appl. Math. 8 (1955), 615-633.

[5] Yosida, K. *Functional Analysis.* Springer; Reprint of the 6th ed. Berlin, Heidelberg, New York 1980 edition (1995).

Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA
E-mail address: marcelo.disconzi@vanderbilt.edu