Holographic renormalization and correlation functions

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Abstract

In this talk we'll derive explicit relations between the $(d+1)$-bulk theory and the d-dimensional QFT. This will be the "AdS-CFT prescription at work".

1 Introduction

Broadly speaking, holography states that a $(d+1)$-dimensional gravitational theory — referred as the **bulk theory** (whose quantities are therefore called bulk quantities, e.g., bulk metric etc) should have a description in terms of a d-dimensional quantum field theory — referred as the **dual theory**. AdS-CFT provides a realization of this idea [1].

In this talk we'll derive explicit relations between the $(d+1)$-bulk theory and the d-dimensional QFT. This will be the "AdS-CFT prescription at work".

Before starting, however, a word of caution: we'll be working on a $(d+1)$-dimensional conformally compact Einstein manifold $\overline{X}$; the dual QFT "lives" on the boundary ∂X of X. This QFT needs not to be a quantum conformal field theory (in fact, the example we'll present is *not* a CFT since it has a conformal anomaly), neither is $\overline{X}$ necessarily an AdS space. However, the "holographic principle":

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$$\text{gravity on } X \leftrightarrow \text{QFT on } \partial X$$

is sometimes also called AdS-CFT correspondence and we might use this jargon as well (although we'll use "holographic principle/correspondence" etc when we want to stress that we don't have a CFT).

2 Correlation functions

Recall that correlation functions (c.f.) are the main objects of interest in QFT since observable quantities can be expressed in terms of them. These are defined as expectation values of time ordered products of operators:

$$\langle \hat{\mathcal{O}}(x_1) \cdots \hat{\mathcal{O}}(x_n) \rangle \tag{1}$$

and can be expressed in terms of path integrals

$$\langle \hat{\mathcal{O}}(x_1) \cdots \hat{\mathcal{O}}(x_n) \rangle = \frac{\int \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) e^{iS} \mathcal{D}\mathcal{O}}{\int e^{iS} \mathcal{D}\mathcal{O}} \tag{2}$$

($\hat{\mathcal{O}}$=quantum field =operator; $\mathcal{O}$=classical field=function, spinor etc).

Correlation functions can be computed by functionally differentiating a generating functional $Z[J]$ and setting the source J equal to zero:

$$Z[J] = \frac{e^{iS+i\int \mathcal{O}J} \mathcal{D}\mathcal{O}}{\int e^{iS} \mathcal{D}\mathcal{O}} \tag{3}$$

$$\langle \hat{\mathcal{O}}(x_1) \cdots \hat{\mathcal{O}}(x_n) \rangle = \frac{1}{i} \frac{\delta}{\delta J(x_1)} \cdots \frac{1}{i} \frac{\delta}{\delta J(x_n)} Z[J] \Big|_{J=0} \tag{4}$$

where the integral $\int \mathcal{O}J$ is over spacetime.

Recall that in general the c.f. diverge and need to be renormalized. Recall, also, that most of the time we are interested in connected Feynman graphs and therefore we want to consider the generating functional

$$W[J] = -i \log Z[J] \tag{5}$$

instead of $Z[J]$. In this case we obtain connected c.f. $\langle \hat{\mathcal{O}}(x_1) \cdots \hat{\mathcal{O}}(x_n) \rangle_{\text{connected}}$ by functionally differentiating $W[J]$. Since in the end of the day we'll use $W[J]$ rather than $Z[J]$, we'll drop the subscript "connected" and write simply $\langle \hat{\mathcal{O}}(x_1) \cdots \hat{\mathcal{O}}(x_n) \rangle$.

3 Witten's prescription

Witten [3] gave a precise formulation of the correspondence initially conjectured by Maldacena [5]. First we recall some definitions:

A metric $\mathring{g}$ on the interior of X^{d+1} of a compact manifold with boundary $\overline{X}^{d+1}$ is said to be **conformally compact** if $\bar{g} = r^2\mathring{g}$ extends continuously (or with some degree of smoothness) as a metric to $\overline{X}$, where r is a **defining function** for ∂X, i.e., $r > 0$ on X and $r = 0,\ dr \neq 0$ on ∂X.

Because of the possible choices for r only the conformal class $[\bar{g}\big|_{\partial X}]$ of $\bar{g}\big|_{\partial X}$ is determined by the original data. $(\partial X, [\bar{g}\big|_{\partial X}])$ is called conformal infinity of $(X, \mathring{g})$; as abuse of language, sometimes ∂X is called the conformal infinity. If $(X, \mathring{g})$ is Einstein then we say that we have a **conformally compact Einstein manifold** (c.c.E.m). See [4] for more details.

From now on we'll always be working with a c.c.E.m. $(X^{d+1}, \mathring{g})$. We'll also assume throughout these notes that the cosmological constant in the Hilbert-Einstein action is negative and we'll, as usually, write the cosmological constant as $\Lambda = -\frac{d(d-1)}{2l^2}$, with $l^2 = 1$ (see below).

In order to state Witten's prescription we'll denote by ϕ general fields on X (in the example we'll work out later the field is the metric). Let $Z_S(\phi_0)$ be the supergravity partition function (or string; $Z_S(\phi_0)$ is always a partition function of a gravitational theory) on X computed with the boundary condition that at infinity ϕ approaches a given field ϕ_0.

The holographic principle/AdS-CFT correspondence states that for every bulk field ϕ there exists a corresponding gauge invariant operator $\hat{\mathcal{O}}_\phi$ on ∂X[1] (recall that "bulk" refers to quantities on X, so "bulk field" simply means a field on X); it also states that ϕ_0 couples to $\mathcal{O}_\phi$: $\int \mathcal{O}_\phi \phi_0$ (and therefore ϕ_0 acts as a source for $\mathcal{O}_\phi$, see below). In this situation Witten's prescription is:

$$Z_S(\phi_0) = \langle e^{-\int_{\partial X} \phi_0 \mathcal{O}_\phi} \rangle_{QFT} \tag{6}$$

which, unwrapping the definitions, reads:

$$\int_{\phi \sim \phi_0} e^{-S_{gr}(\phi)} \mathcal{D}\phi = \frac{\int e^{-S(\mathcal{O}_\phi)} e^{-\int_{\partial X} \phi_0 \mathcal{O}_\phi} \mathcal{D}\mathcal{O}_\phi}{\int e^{-S(\mathcal{O}_\phi)} \mathcal{D}\mathcal{O}_\phi} \tag{7}$$

[1] The subscript ϕ on $\hat{\mathcal{O}}_\phi$ is simply to remind us that this is the operator corresponding, via the holographic principle, to the bulk field ϕ; it doesn't mean that $\hat{\mathcal{O}}_\phi$ depends on ϕ.

where $\int_{\phi \sim \phi_0} e^{-S_{gr}(\phi)} \mathcal{D}\phi$ means the path integral over fields on X which approach ϕ_0 at infinity, $S_{gr}(\phi)$ is the classical action for the gravitational theory (supergravity, strings etc) on X and $S(\mathcal{O}_\phi)$ is the classical action for the QFT on ∂X. As the reader probably noticed, all the quantities are Wick rotated.

Notice that the right hand side of (6) (or equivalently (7)) is exactly the generating functional for c.f. of $\mathcal{O}_\phi$, with ϕ_0 as source (compare with (3) where ϕ_0 plays the role of J). Hence, functionally differentiating $\langle e^{-\int_{\partial X} \phi_0 \mathcal{O}_\phi} \rangle_{QFT}$ with respect to ϕ_0 (which is how we compute c.f.) is equivalent to functionally differentiating $Z_S(\phi_0)$ w.r.t. ϕ_0. Here is the "magic" of the holographic principle: the c.f. of the QFT on ∂X can be obtained from the partition function of the gravitational theory on X. And this is more than just a clever computational trick: since a QFT is completely determined by its c.f., the QFT on ∂X is completely determined by the (in principle, completely different) gravitational theory in X.

We'll assume from now on that this correspondence is true (remember that the holographic principle/AdS-CFT correspondence is a conjecture, although there are compelling arguments in its favor). Following the jargon, we'll talk about the "gravitational side" and the "QFT side" or yet the "CFT side" of the correspondence.

In a lot of situations we are interested in the low energy limit on the gravitational side. More precisely, we suppose a regime where on the bulk side the following holds: (i) geometry is a good description, i.e., large radius approximation (by radius we mean the parameter l which appears in the metric, we'll be working in units such that $l = 1$ throughout these notes) and (ii) quantum effects are negligible (so that we have a good saddle point approximation). These are correlated on the QFT side with (i) large 't Hooft coupling (strongly coupled gauge theory only) and (ii) large N, where N comes from the gauge group $SU(N)$ — see [3, 5] for a more detailed discussion.

In this situation we can compute $Z_S(\phi_0)$ by a saddle point approximation; as usual only the classical contribution survives:

$$Z_S(\phi_0)\Big|_{\text{low energy}} = e^{-S_{os}(\phi_0)} \tag{8}$$

where $S_{os}(\phi_0)$ is the on-shell classical action, i.e., it is the action S_{gr} evaluated at the solution ϕ of the Euler-Lagrange equations with boundary condition $\phi = \phi_0$ at infinity.

Of course, there are various issues here: we mention "the" solution, but we might not have uniqueness and in that case we have to sum over the solutions ϕ which satisfy the given boundary condition. Also, the action itself might diverge, i.e., $S_{os}(\phi_0) = \pm\infty$ and in that case some sort of renormalization needs to be implemented (this will actually be the case in the example we'll develop later). Finally, it might be that the low energy limit on the gravitational side is not a good approximation. In that case quantum corrections need to be added. We'll not deal with this situation here, see [3] and [5] for details.

It is important to stress, however, that a low energy limit on the gravitational side does not mean a low energy limit on the QFT side.

Putting together formulas (7) and (8) we get:

$$e^{-S_{os}(\phi_0)} = Z_{QFT}[\phi_0] \tag{9}$$

where, as we pointed out before, $Z_{QFT}[\phi_0]$=right hand side of (7) is the generating functional of c.f. on the QFT side. Taking log:

$$S_{os}(\phi_0) = -\log Z_{QFT}[\phi_0] = -W_{QFT}[\phi_0] \tag{10}$$

where $W_{QFT}[\phi_0]$ is the generating functional for connected diagrams.

As said before, the c.f. are computed by differentiating $W_{QFT}[\phi_0]$ w.r.t. ϕ_0. From (10) we can then compute the c.f. by differentiating the on-shell action:

$$\langle\hat{\mathcal{O}}(x)\rangle = \frac{\delta}{\delta\phi_0(x)} S_{os}(\phi_0)\Big|_{\phi_0=0} \tag{11}$$

$$\langle\hat{\mathcal{O}}(x_1)\cdots\hat{\mathcal{O}}(x_n)\rangle = (-1)^{n+1}\frac{\delta}{\delta\phi_0(x_1)}\cdots\frac{\delta}{\delta\phi_0(x_n)} S_{os}(\phi_0)\Big|_{\phi_0=0} \tag{12}$$

(the factor $(-1)^{n+1}$ appears instead of a power of i because the quantities are Wick rotated).

4 Fefferman-Graham expansion and volume renormalization

The reference for this section is [4]. The tools presented here will be useful to compute the on-shell action.

Let $(X^{d+1}, \mathring{g})$ be a c.c.E.m, we denote $\bar{g} = r^2\mathring{g}$. It's always possible to choose r such that $|dr|^2_{\bar{g}} = 1$ (indeed, start with any defining function r_0: $\bar{g}_0 = r_0^2\mathring{g}$, conformally change the metric $\bar{g} = e^{2\omega}\bar{g}_0$, compute $|dr|^2_{\bar{g}}$ and set it equal to one, this gives a non-characteristic first order PDE for ω which can therefore be solved).

Denote $\partial X = M$. A defining function determines for some $\epsilon > 0$ an identification of $M \times [0, \epsilon)$ with a neighborhood of M in $\overline{X}$: $(p, \lambda) \in M \times [0, \epsilon)$ corresponds to the point obtained by following the integral curve of $\nabla_{\bar{g}} r$ emanating from p for λ units of time. For r such that $|dr|^2_{\bar{g}} = 1$, the λ coordinate is just r and $\nabla_{\bar{g}} r$ is orthogonal to the slices $M \times \{\lambda\}$. Hence, identifying λ with r on $M \times [0, \epsilon)$ the metric $\bar{g}$ takes the form

$$\bar{g} = dr^2 + g_r \tag{13}$$

where g_r is a 1-parameter family of metrics on M. Then

$$\mathring{g} = \frac{1}{r^2}(dr^2 + g_r) \tag{14}$$

Recall that $\mathring{g}$ satisfies Einstein equations: $\mathrm{Ric}(\mathring{g}) + d \cdot \mathring{g} = 0$ (d=dimension of M). Decomposing the tensor $\mathrm{Ric}(\mathring{g}) + d \cdot \mathring{g}$ into components w.r.t. the product structure $M \times [0, \epsilon)$ gives that the vanishing of the component with both indices in M is given by

$$r g''_{ij} + (1-d) g'_{ij} - g_{ij} g^{kl} g'_{kl} - r g^{kl} g'_{ik} g'_{jl} + \frac{r}{2} g^{kl} g'_{kl} g'_{ij} - 2r R_{ij} = 0 \tag{15}$$

where g_{ij} are the components of g_r on M, $'$ denotes ∂_r and R_{ij} is the Ricci tensor for g_r with r fixed. The above equation can be written in an index-free fashion as

$$r g''_r + (1-d) g'_r - 2K g_r - 2r\,\mathrm{Ric}(g_r) + rK g'_r + r(g'_r)^2 = 0 \tag{16}$$

where K is the mean curvature.

We want to expand g_r near the boundary:

$$g_r = g_{(0)} + r g_{(1)} + r^2 g_{(2)} + \dots \tag{17}$$

The coefficients of this expansion can be calculated by differentiating $(\nu - 1)$ times expression (15) w.r.t. r and setting $r = 0$:

$$(\nu - d)\partial_r^\nu g_{ij}|_{r=0} - g^{kl}(\partial_r^\nu g_{kl}) g_{ij}|_{r=0} = (\text{terms involving } \partial_r^\mu g_{ij},\ \mu < \nu) \tag{18}$$

The odd coefficients of order less than d all vanish, and equation (18) also completely determines $\partial_r^\nu g_{ij}|_{r=0}$ as long as $\nu < d$. When $\nu = d$ we have that if d is odd the right hand side of (18) vanishes by parity considerations and then $g^{kl}(\partial_r^d g_{kl})g_{ij}|_{r=0} = 0$, so the trace free part of $\partial^d g_{kl}$ can be chosen arbitrarily. If d is even, the right hand side of 18 might have non-vanishing trace free part, forcing the inclusion of a $r^d \log r$ term in order to make the expansion consistent — the term $r^d \log r$ has a trace-free coefficient. The trace of the r^d coefficient is determined but not its trace-free part.

Summarizing:

d even:

$$g_r = g_{(0)} + g_{(2)}r^2 + (\text{even powers}) + hr^d \log r + g_{(d)}r^d + \dots \tag{19}$$

where the $g_{(j)}$ are locally formally determined for $0 < j \leq d-2$, h is locally formally determined and trace-free and the trace of $g_{(d)}$ is locally determined. $g_{(0)}$ and the trace-free part of $g_{(d)}$ are formally undetermined.

d odd:

$$g_r = g_{(0)} + g_{(2)}r^2 + (\text{even powers}) + g_{(d-1)}r^{d-1} + g_{(d)}r^d + \dots \tag{20}$$

where the $g_{(j)}$ are locally formally determined for $0 < j \leq d-1$, $g_{(0)}$ and the trace-free part of $g_{(d)}$ are formally undetermined but $g_{(d)}$ is trace-free.

Of course, there are convergence issues etc. We are treating the series purely formally.

So, to calculate the coefficients we have to carry out the above differentiations and keep track of the terms. For example, for $d = 2$:

$$h = 0 \tag{21}$$

$$g_{(0)}^{ij} g_{(2)ij} = -\frac{1}{2}R \tag{22}$$

Notice that $g_{(0)}$ corresponds to a representative of the conformal infinity of $(X, \mathring{g})$. In particular, when we are solving Einstein's equations with condition $g = \tilde{g}$ at infinity we have $g_{(0)} = \tilde{g}$.

We notice for further use that

$$\left(\frac{\det g_r}{\det g_{(0)}}\right)^{\frac{1}{2}} = 1 + v_{(2)}r^2 + (\text{even powers}) + v_{(d)}r^d + \dots \tag{23}$$

where all $v_{(j)}$'s are locally determined functions on M.

5 Holographic renormalization and correlation functions

The main reference here is [1].

Conformally compact Einstein manifolds can be thought of as asymptotically AdS in the following sense. The curvature of $(X^{d+1}, \mathring{g})$ near the conformal infinity (i.e., near the boundary) takes the form[2]:

$$R_{ijkl}(\mathring{g}) = |dr|^2_{\bar{g}}(\mathring{g}_{ik}\mathring{g}_{lj} - \mathring{g}_{kj}\mathring{g}_{li}) + O(r^{-3}) \tag{24}$$

For $|dr|^2_{\bar{g}} = 1$

$$R_{ijkl}(\mathring{g}) = \mathring{g}_{ik}\mathring{g}_{lj} - \mathring{g}_{kj}\mathring{g}_{li} + O(r^{-3}) \tag{25}$$

Notice that $\mathring{g}_{ik}\mathring{g}_{lj} - \mathring{g}_{kj}\mathring{g}_{li}$ is of order $O(r^{-4})$, so it dominates near the boundary and $\mathring{g}_{ik}\mathring{g}_{lj} - \mathring{g}_{kj}\mathring{g}_{li}$ is exactly the expression for the curvature of AdS spaces. Hence we might define an asymptotically AdS space as a c.c.E.m. [2].

We want to use formulas (10) and (12) with the bulk field being the metric. In this case $S_{gr}(\phi) = S_{gr}(\mathring{g})$ is the Hilbert-Einstein action (as always, we imagine that some boundary condition $\mathring{g} = g_{(0)}$ is prescribed at infinity):

$$S_{os}(g_{(0)}) = \int_X (R + 2\Lambda) + \int_{\partial X} 2K \tag{26}$$

Of course, (26) is ill-defined: Einstein's equation imply that R is constant, hence $\int_X$ is proportional to the volume of X, which is infinity. Also, there is the question of what exactly the term $\int_{\partial X}$ is supposed to mean with the boundary at infinity.

From the point of view of (10) and (12), this might not be a surprise since the c.f. obtained from W_{QFT} are known to diverge.

Hence, we need to apply some renormalization procedure to render $S_{os}(g_{(0)})$ finite. After doing so we can then apply the AdS-CFT prescription to compute correlation functions. The point to be stressed here is that by carrying out the suitable renormalization on the gravitational side equations (10) and (12) will automatically yield *renormalized* c.f.

First, it's going to be more convenient to rewrite (14) in coordinates $\rho = r^2$; in this case:

$$\mathring{g} = \frac{1}{r^2}(dr^2 + g_r) = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_\rho \tag{27}$$

[2]We use the convention of [2]: $R_{ijk}{}^{l} = \partial_i \Gamma^l_{jk} + \Gamma^l_{im}\Gamma^m_{jk} - \mu \leftrightarrow \nu,\ R_{ij} = R_{ikj}{}^{k}$.

Then (20) becomes:

$$g_\rho = g_{(0)} + \rho g_{(2)} + \cdots + \rho^{d/2} g_{(d)} + h\rho^{d/2}\log\rho + \ldots \tag{28}$$

(we'll write everything for d even, similar formulas hold for d odd). And (23) becomes

$$\Big(\frac{\det g_\rho}{\det g_{(0)}}\Big)^{\frac{1}{2}} = 1 + v_{(2)}\rho + \cdots + v_{(d)}\rho^{d/2} + \ldots \tag{29}$$

(recall that the coefficients $v_{(j)}$ are known, i.e., they are locally determined functions on M).

We start by regularizing the on-shell Hilbert-Einstein action, i.e., we introduce a cutoff which prevents us from integrating all way up to the boundary:

$$S_{os,reg}(g_{(0)}) = \int_{\rho\geq\epsilon} (R(\mathring{g}) + 2\Lambda)\sqrt{\mathring{g}}dxd\rho - \int_{\rho=\epsilon} 2K\sqrt{\gamma}dxd\rho \tag{30}$$

where γ is the metric induced on $\{\rho = \epsilon\}$ (and $K = K(\gamma)$) (the minus sign comes from the fact that the boundary corresponds to $\rho = \epsilon$ =lower limit of integration). Notice that from $\mathring{g} = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}g_\rho$ we have that $\gamma = \frac{1}{\rho}g_\rho$.

Since $\mathring{g}$ satisfies Einstein's equations we get $R = \frac{2\Lambda(d+1)}{1-d}$ (recall that $\dim X = d+1$) and using $\Lambda = -\frac{d(d-1)}{2}$ we get $R + 2\Lambda = 2d$. Using also

$$\det\mathring{g} = \frac{1}{4\rho^2}\det(\frac{1}{\rho}g_\rho) = \frac{1}{4\rho^2}\frac{1}{\rho^d}\det(g_\rho) \tag{31}$$

we get

$$\int_{\rho\geq\epsilon} = \int_{\rho=\epsilon}\Big(\int_\epsilon \frac{d}{\rho^{\frac{d}{2}+1}}\sqrt{\det g_\rho}d\rho\Big)dx \tag{32}$$

(here $\int_\epsilon$ means $\int_\epsilon^C$, where C is some appropriate constant).

For the boundary integral, we use that $\gamma = \frac{1}{\rho}g_\rho$, $k_{ij} = \frac{1}{2}\partial_r\gamma_{ij} = \frac{1}{2}\frac{\partial\gamma_{ij}}{\partial\rho}\frac{\partial\rho}{\partial r} = \sqrt{\rho}\partial_\rho(\frac{1}{\rho}g_{ij}(x,\rho))$ and $\gamma^{ij} = \rho g^{ij}(x,\rho)$ to obtain the following expression for the integral of $K = \gamma^{ij}k_{ij}$:

$$-\int_{\rho=\epsilon} = \int_{\rho=\epsilon}\frac{1}{\rho^{d/2}}\Big(-2d\sqrt{\det g_\rho} + 4\rho\partial_\rho\sqrt{\det g_\rho}\Big)\Big|_{\rho=\epsilon}dx \tag{33}$$

where we also have used the standard formulas for derivative of the determinant:

$$\frac{1}{\det M}\frac{d}{dt}detM = \mathrm{tr}\left(\frac{dM}{dt}M^{-1}\right) \tag{34}$$

$$\delta\sqrt{\det g} = \frac{1}{2}\sqrt{\det g}g^{ij}\delta g_{ij} \tag{35}$$

which in our case allows us to express $g^{ij}(x,\rho)\partial_\rho g_{ij}(x,\rho)$ in terms of $\partial_\rho\sqrt{\det g_\rho}$:

$$\partial_\rho\sqrt{\det g_\rho} = \frac{1}{2}\sqrt{\det g}g^{ij}\partial_\rho g_{ij} \tag{36}$$

Putting all together

$$S_{on,reg}(g_{(0)}) = \int\Big[\int_\epsilon \frac{d}{\rho^{\frac{d}{2}+1}}\sqrt{\det g_\rho}d\rho + \frac{1}{\rho^{d/2}}\Big(-2d\sqrt{\det g_\rho} + 4\rho\partial_\rho\sqrt{\det g_\rho}\Big)\Big|_{\rho=\epsilon}\Big]dx \tag{37}$$

Using now (29) we find that the divergences appear as $\frac{1}{\epsilon^k}$ poles plus a logarithmic divergence:

$$S_{on,reg}(g_{(0)}) = \int\sqrt{\det g_{(0)}}\Big(a_{(0)}\epsilon^{-\frac{d}{2}} + a_{(2)}\epsilon^{-\frac{d}{2}+1} + \cdots + a_{(d-2)}\epsilon^{-1} + a_{(d)}\log\epsilon\Big)dx + O(\epsilon^0) \tag{38}$$

The $a_{(j)}$'s come from the $v_{(j)}$'s and hence are known quantities. Now we renormalize the action by subtracting the divergent terms and removing the cutoff:

$$S_{on,ren}(g_{(0)}) = \lim_{\epsilon\to 0}\Big[S_{on,reg}(g_{(0)}) - \int\sqrt{\det g_{(0)}}\Big(a_{(0)}\epsilon^{-\frac{d}{2}} + a_{(2)}\epsilon^{-\frac{d}{2}+1} + \cdots + a_{(d-2)}\epsilon^{-1} + a_{(d)}\log\epsilon\Big)dx\Big] \tag{39}$$

It seems that now we would be in a condition to use formulas (10) and (12) to compute c.f. on the QFT side. There is, however, a technical issue. We changed the on-shell action via the renormalization procedure. It's not obvious that (12) should hold with $S_{on,ren}(g_{(0)})$ in place of $S_{on}(g_{(0)})$. In fact, it won't. For the 1-point function, which is the case we'll be interested here, (12) needs to be modified to

$$\langle\hat{\mathcal{O}}_{ij}(x)\rangle = \frac{2}{\sqrt{\det g_{(0)}(x)}}\frac{\delta S_{os,ren}(g_{(0)})}{\delta g^{ij}_{(0)}(x)}\Big|_{g_{(0)}=0} \tag{40}$$

see [2] for more details.

We can ask which operator $\hat{\mathcal{O}}$ is the dual of the bulk field $\mathring{g}$. Recall that according the the holographic principle $\hat{\mathcal{O}}$ is the quantum field corresponding to the classical field $\mathcal{O}$ which couples to $g_{(0)}$ in the classical action — which turns out to be the stress energy tensor (notice that this is the stress-energy tensor of the dual theory, it's *not* the bulk stress energy tensor).

Hence

$$\langle \hat{T}_{ij}(x) \rangle = \frac{2}{\sqrt{\det g_{(0)}(x)}} \frac{\delta S_{os,ren}(g_{(0)})}{\delta g^{ij}_{(0)}(x)} \Big|_{g_{(0)}=0} \tag{41}$$

Using $\gamma = \frac{1}{\rho} g_\rho$ the above equation can be expressed as

$$\begin{aligned} \langle \hat{T}_{ij}(x) \rangle = \frac{2}{\sqrt{\det g_{(0)}(x)}} \frac{\delta S_{os,ren}(g_{(0)})}{\delta g^{ij}_{(0)}(x)} \Big|_{g_{(0)}=0} &= \lim_{\epsilon \to 0} \frac{2}{\sqrt{\det g(x,\epsilon)}} \frac{\delta S_{os,ren}(g_{(0)})}{\delta g^{ij}(x,\epsilon)} \\ &= \lim_{\epsilon \to 0} \Big(\frac{1}{\epsilon^{\frac{d}{2}-1}} T_{ij}(\gamma) \Big) \end{aligned} \tag{42}$$

Where $T_{ij}(\gamma)$ is the stress-energy tensor of the theory at $\rho = \epsilon$ described by the action (39) but before the limit is taken.

$T_{ij}(\gamma)$ has two contributions:

$$T_{ij}(\gamma) = T^{reg}_{ij} + T^{ct}_{ij} \tag{43}$$

where T^{reg}_{ij} comes from the regularized action (30) and T^{ct}_{ij} comes from the counter-terms. T^{reg}_{ij} can be computed from (37) by varying with respect to the induced metric (notice however that $T^{reg}_{ij} = -K_{ij} + K\gamma_{ij}$); the answer is

$$T^{reg}_{ij} = \partial_\epsilon g_{ij}(x,\epsilon) - g_{ij}(x,\epsilon) g^{kl}(x,\epsilon) \partial_\epsilon g_{kl}(x,\epsilon) + \frac{1-d}{\epsilon} g_{ij}(x,\epsilon) \tag{44}$$

The contribution T^{ct}_{ij} has to be computed from

$$-\int \sqrt{\det g_{(0)}} \Big(a_{(0)} \epsilon^{-\frac{d}{2}} + a_{(2)} \epsilon^{-\frac{d}{2}+1} + \cdots + a_{(d-2)} \epsilon^{-1} + a_{(d)} \log \epsilon \Big) \tag{45}$$

First we need to rewrite all quantities in terms of the induced metric — this is done by inverting the relation between γ and $g_{(0)}$ perturbatively in ϵ; then we vary γ to get the stress-energy tensor and finally write everything back in terms of $g_{ij}(x,\epsilon)$. These computations and the general expressions derived from them can be found in [1]. In dimension two, however, the computations are simpler and the formulas more friendly, and in this case the contribution of the counter-terms is:

$$T_{ij}^{ct} = -\gamma_{ij} = -\frac{1}{\epsilon}g_{ij}(x,\rho) = -\frac{1}{\epsilon}(g_{(0)ij} + \epsilon g_{(2)ij} + \dots) = -\frac{g_{(0)ij}}{\epsilon} - g_{(2)ij} + O(\epsilon) \tag{46}$$

(we don't have a log divergence in dimension two; see the explicit expressions in [1]).

We assume $d = 2$ from now on. In this case, using $g_{ij}(x,\epsilon) = g_{(0)ij} + \epsilon g_{(2)ij} + \dots$ (recall that $h = 0$ in two dimensions) expression (44) becomes:

$$T_{ij}^{reg} = 2g_{(2)ij} - g_{(0)ij}g_{(2)}^{kl}g_{(2)kl} + \frac{g_{(0)ij}}{\epsilon} + O(\epsilon) \tag{47}$$

Putting all these in (43) and using (42) we finally obtain

$$\langle \hat{T}_{ij} \rangle = g_{(2)ij} - g_{(0)ij}g_{(2)}^{kl}g_{(2)kl} \tag{48}$$

Notice that the divergent terms cancel, as expected.

Therefore, we have successfully derived an expression for the one-point function of the QFT on the boundary in terms of data coming from the bulk. Recall that $g_{(0)}$ and $g_{(2)}$ are unknowns in the Fefferman-Graham expansion. Suppose, however, that we *completely* solved the bulk theory in such way that $g_{(0)ij}$ and $g_{(d)ij}$ are known. Then formula (48) tells us how to find the one-point function of the QFT on the boundary. Moreover, as pointed out before, the one-point function given in (48) is a *renormalized* one-point function — the renormalization usually applied to the c.f. was carried out on the gravitational side. The procedure of obtaining renormalized QFT c.f. from the renormalized gravitational on-shell action is called **Holographic renormalization**.

From (48) we can also compute the conformal anomaly of the boundary QFT; we find

$$\text{trace of } \langle \hat{T}_{ij} \rangle = -R = -\frac{3lR}{48\pi G_N} \tag{49}$$

where in the last step we restored all constants omitted in the action. This expression for the conformal anomaly was calculated on [6] using a different approach; we see that the holographic principle gave us the correct answer.

It's interesting to noticing that we can use (48) the other way around: given a metric $g_{(0)}$ on the boundary, can we reconstruct (at least locally) the spacetime on the interior? From the Fefferman-Graham expansion we see that only $g_{(0)}$ is not enough, we need also $g_{(2)}$. But if we solved the the boundary theory, then we have the one-point function and can use (48) to find $g_{(2)}$ and then reconstruct the bulk metric (at least locally) from the Fefferman-Graham expansion (this is in fact the point of view of [1]).

Needless to say, the above reasoning illustrated with equation (48) applies in general dimension.

Finally, it's important to stress that the ideas presented here are quite general and can be applied to other fields than the metric; see [2].

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