Quasi-linear wave equations

We are interested in the Cauchy problem for a function $u : \mathbb{R}^{n+1} \to \mathbb{R}^N$ of the form

$$g^{\mu\nu}(u, \partial u)\, \partial_\mu \partial_\nu u = f(u, \partial u).$$

We denote $\partial u = (\partial_0 u, \partial_1 u, \dots, \partial_n u)$ and

$$g^{\mu\nu}(u, \partial u) = g^{\mu\nu}(t, x, u(t,x), \partial_0 u(t,x), \dots, \partial_n u(t,x))$$

and similarly for f. Here, $(t,x) = (x^0, x^1, \dots, x^n)$ are Cartesian coordinates in $\mathbb{R}^{n+1}$. We will be interested in the case when, for suitable u, we have a Lorentzian metric $g_{\mu\nu}(u, \partial u)$ with inverse $g^{\mu\nu}(u, \partial u)$. We sometimes write simply $g[u]$, g_u or g to simplify the notation.

Einstein's equations in suitable coordinates have the above form. In that case u will be the vector whose entries are the components of g themselves. We are primarily interested in $n = 3$ but we keep n general for now.

Basic assumptions and notation

We consider a C^k-map, k large,

$$g : \mathbb{R}^{n+1} \times \underbrace{\mathbb{R}^N \times \dots \times \mathbb{R}^N}_{(n+2)\text{ times}} \longrightarrow L_{n+1}$$

where L_{n+1} are the $(n+1) \times (n+1)$ Lorentz matrices in $\mathbb{R}^{n+1}$ with negative (00)-entry. We write $g = (g_{\mu\nu})$ and $g^{-1} = (g^{\mu\nu})$. We assume that for every multi-index $\alpha = (\alpha_0, \dots, \alpha_{n+1+(n+2)N})$, $|\alpha| \le k$, and every compact interval $I = [T_1, T_2]$, there exists a continuous increasing $J_{\alpha, I} : \mathbb{R} \to \mathbb{R}$ such that

$$|(\partial^\alpha g_{\mu\nu})(t, x, \xi)| \le J_{\alpha, I}(|\xi|).$$

We also assume that for each $I = [T_1, T_2]$ it holds that

$$g_{00} \le -a_I, \quad (g_{ij}) \ge a_I, \quad \sum_{\mu,\nu=0}^{n} |g_{\mu\nu}| \le a_I, \quad \text{for some } a_I > 0.$$

For the function f, we similarly assume

$$|\partial^\alpha f(t, x, \xi)| \le J_{\alpha, I}(|\xi|).$$

We also assume that $f(t,x,v)$ has the property that for any $[T_1,T_2]$, $f(t,x,v)=0$ for $t\in[T_1,T_2]$ and x outside some compact set (possibly depending on $[T_1,T_2]$)

We will use the following convention. For g and f as above and $I=[T_1,T_2]$, we denote $F_I = F_I(g,f)$ a continuous non-negative map on $\mathbb{R}$ such that $F_I \leq F_{I'}$ if $I\subseteq I'$. If it is constant we write G_I. F_I and G_I can change from line to line.

The Sobolev norm on constant t-slices, i.e. on $\{x^0 = \text{constant}\} = \{t=\text{constant}\}$ will be denoted $\|\cdot\|_s$. In particular, $\|\cdot\|_0$ denotes the L^2 norm.

Energy estimate

In what follows, multi-indices are spatial unless stated otherwise. Set, with $s > \frac{n}{2}+1$,

$$\mathcal{M}_s(u) \equiv \mathcal{M}_s(u)(t) = \|u(t,\cdot)\|_{s+1} + \|\partial_t u(t,\cdot)\|_s$$

$$\mathcal{N}(u) \equiv \mathcal{N}(u)(t) = \sum_{|\alpha|+j\le 2} \sup_{x\in\mathbb{R}^n} |\partial^\alpha \partial_t^j u(t,x)|$$

Let $v \in C^\infty(\mathbb{R}^{n+1},\mathbb{R}^N)$ have the property that for any $[T_1,T_2]$ we have $v(t,x)=0$ for $t\in[T_1,T_2]$ and x outside some compact set (possibly depending on $[T_1,T_2]$). Let u solve the linear problem:

$$g_v^{\mu\nu}\partial_\mu\partial_\nu u = f_v,$$
$$u(0) = u_0,$$
$$\partial_t u(0) = u_1,$$

where u_0, u_1 are smooth. Then

$$\mathcal{M}_s(u) \le C_I\, \mathcal{M}_s(u)(0)$$
$$+ \int_0^t \Big(C_I + F_I(\mathcal{N}(v))\big((1+\mathcal{N}(u))\,\mathcal{M}_s(v) + \mathcal{M}_s(u)\big)\Big).$$

Moreover, set:

$$E_s(\sigma,u) = E_s(\sigma,u)(t) = \frac{1}{2}\sum_{|\alpha|\leq s}\int_{\mathbb{T}^3}\Big(-g_\sigma^{00}|\partial^\alpha\partial_t u|^2 + g_\sigma^{ij}\partial^\alpha\partial_i u\cdot\partial^\alpha\partial_j u$$

$$+ |\partial^\alpha u|^2\Big).$$

Then, for $t \in I = [0,T]$

$$\partial_t E_s(\sigma,u) \lesssim C_I + F_I(\mathcal{N}(\sigma),\mathcal{N}(u))\big(\mathcal{M}_s^2(\sigma) + E_s(\sigma,u)\big).$$

We proceed to establish these inequalities. Note that we can assume without loss of generality that $g_\sigma^{00} = -1$.

We first consider $s=0$ and compute

$$\partial_t E_0 = \int_{\mathbb{T}^3}\Big(-g_\sigma^{00}\partial_t u\cdot\partial_t^2 u - \frac{1}{2}\partial_t g_\sigma^{00}|\partial_t u|^2 + g_\sigma^{ij}\partial_i u\cdot\partial_t\partial_j u$$

$$+ \frac{1}{2}\partial_t g_\sigma^{ij}\partial_i u\cdot\partial_j u + u\cdot\partial_t u\Big)$$

Using
$$-g_\sigma^{00}\partial_t^2 u = -g_\sigma^{\mu\nu}\partial_\mu\partial_\nu u + 2g_\sigma^{0i}\partial_0\partial_i u + g_\sigma^{ij}\partial_i\partial_j u$$
$$= -f_\sigma + 2g_\sigma^{0i}\partial_0\partial_i u + g_\sigma^{ij}\partial_i\partial_j u$$

we have

$$= \int_{\mathbb{R}^n} \Big(-f_v \cdot \partial_t u - \frac{1}{2} \partial_t g_v^{00} |\partial_t u|^2 + 2 g_v^{0i} \partial_0 \partial_i u \cdot \partial_t u + g_v^{ij} \partial_i \partial_j u \cdot \partial_t u$$

$$+ g_v^{ij} \partial_i u \cdot \partial_t \partial_j u + \frac{1}{2} \partial_t g_v^{ij} \partial_i u \cdot \partial_j u + u \cdot \partial_t u \Big)$$

But $\displaystyle\int_{\mathbb{R}^n} 2 g_v^{0i} \partial_0 \partial_i u \cdot \partial_t u = \int_{\mathbb{R}^n} g_v^{0i} \partial_i |\partial_t u|^2 = - \int_{\mathbb{R}^n} \partial_i g_v^{0i} |\partial_t u|^2$

$$\int_{\mathbb{R}^n} g_v^{ij} \partial_i \partial_j u \cdot \partial_t u = - \underbrace{\int_{\mathbb{R}^n} g_v^{ij} \partial_i u \cdot \partial_j \partial_t u}_{\text{cancels with similar term}} - \int_{\mathbb{R}^n} \partial_j g_v^{ij} \partial_i u \, \partial_t u \quad , \text{ so}$$

$$\partial_t E_0 = \int_{\mathbb{R}^n} \Big(-f_v \cdot \partial_t u - \frac{1}{2} \partial_t g_v^{00} |\partial_t u|^2 - \partial_i g_v^{0i} |\partial_t u|^2 - \partial_j g_v^{ij} \partial_i u \cdot \partial_j u$$

$$+ \frac{1}{2} \partial_t g_v^{ij} \partial_i u \cdot \partial_j u + u \cdot \partial_t u \Big)$$

Now we have: $\partial_t g_v^{00} = 0$

$$|\partial_i g_v^{0i}| = |\partial_i (g^{0i}(v, \partial v))| \leq \sum_{|\alpha| \leq 1} |\partial^\alpha g^{0i}(v, \partial v)| \, |\partial_i \partial v|$$

$$\leq J_{\alpha, I}(|v|, |\partial v|) \, N(v) \leq F_I(N(v))$$

and similarly for the other ∂g terms. Note that $\partial_i g \sim \partial_i \partial v \sim \partial_i \partial_j v, \partial_i \partial_t v$, $\partial_t g \sim \partial_t \partial v \sim \partial_t \partial_i v, \partial_t^2 v$, all terms included in $m(v)$.

We conclude:

$$\partial_t E_0 \lesssim \|f_v\|_0 \|\partial_t u\|_0 + F_I(\mathcal{N}(v))\left(\|\partial_t u\|_0^2 + \|\bar\partial u\|_0^2\right) + \|u\|_0 \|\partial_t u\|_0$$
$$\lesssim \|f_v\|_0 \|\partial_t u\|_0 + F_I(\mathcal{N}(v))\left(\|\partial_t u\|_0^2 + \|\bar\partial u\|_0^2 + \|u\|_0^2\right)$$

where $\bar\partial$ represents spatial derivatives. Using the bounds $g^{ij} \geq a_I$ etc., we have

$$\partial_t E_0 \lesssim \|f_v\|_0 E_0^{1/2} + F_I(\mathcal{N}(v)) E_0 .$$

f_v can be written as $f(t,x,0,0) + f(t,x,v,\partial v) - f(t,x,0,0)$.
Recalling that $f(t,x,0,0)$ has compact support in x:

$$\|f(t,x,0,0)\|_0 \lesssim C_I .$$

Write $f(t,x,v,\partial v) - f(t,x,0,0) = f(t,x,v,\partial v) - f(t,x,v,0)$
$+ f(t,x,v,0) - f(t,x,0,0)$, and

$$f(t,x,v,\partial v) - f(t,x,v,0) = \int_0^1 \partial_\tau f(t,x,v,\tau\partial v)\, d\tau$$

$$\lesssim J_{\alpha,I}(|v|,|\partial v|)\,|\partial v| \lesssim F_I(\mathcal{N}(v))$$

and similarly for the v component, so

$$\|f_v\|_0 \lesssim C_I + F_I(\mathcal{N}(v)) .$$

Thus

$$\partial_t E_0 \leq \left(C_I + F_I(\mathcal{N}(v))\right) E_0^{1/2} + F_I(\mathcal{N}(v)) E_0$$

Use $\quad C_I E_0^{1/2} \leq \frac{1}{2} C_I^2 + \frac{1}{2} E_0$

$$F_I(\mathcal{N}(v)) E_0^{1/2} \leq \frac{1}{2} + \underbrace{F_I^2(\mathcal{N}(v))}_{= F_I(\mathcal{N}(v))} E_0 \qquad \text{to get}$$

$$\partial_t E_0 \leq C_I + F_I(\mathcal{N}(v)) E_0 ,$$

which gives the inequality for $\partial_t E_s$ when $s=0$.

We next notice that under our assumptions

$$\frac{1}{C_I} E_s^{1/2}(v,w) \leq \mathcal{M}_s(w) \leq C_I E_s^{1/2}(v,w)$$

Dividing the inequality on the top of this page by $E_0^{1/2}$:

$$\underbrace{\frac{\partial_t E_0}{E_0^{1/2}}}_{= \partial_t E_0^{1/2}} \leq C_I + F_I(\mathcal{N}(v)) + F_I(\mathcal{N}(v)) \underbrace{E_0^{1/2}}_{\leq C_I \mathcal{M}_0(w)}$$

Integrating in time gives the inequality for $\mathcal{M}_s$ with $s=0$.

Next, apply ∂^α to the equation

$$\partial^\alpha (g^{\mu\nu}\partial_\mu\partial_\nu u) = \partial^\alpha f_\sigma$$

$$\shortparallel$$

$$g^{\mu\nu}\partial_\mu\partial_\nu\partial^\alpha u + [\partial^\alpha, g^{\mu\nu}\partial_\mu\partial_\nu]u$$

so

$$g^{\mu\nu}\partial_\mu\partial_\nu\partial^\alpha u = \partial^\alpha f_\sigma + [g^{\mu\nu}\partial_\mu\partial_\nu, \partial^\alpha]u.$$

We now apply the inequality

$$\partial_t E_0 \lesssim \|f_\sigma\|_0 E_0^{1/2} + F_I(\mathcal{N}(\sigma))E_0$$

that we obtain before with $u \mapsto \partial^\alpha u$, $f_\sigma \mapsto \partial^\alpha f_\sigma + [g^{\mu\nu}\partial_\mu\partial_\nu, \partial^\alpha]$ and sum over $|\alpha| \leq s$ to obtain:

$$\partial_t E_s \lesssim \|f_\sigma\|_s E_s^{1/2} + F_I(\mathcal{N}(\sigma))E_s + \sum_{|\alpha|\leq s} \|[g_\sigma^{\mu\nu}\partial_\mu\partial_\nu, \partial^\alpha]u\|_0 E_s^{1/2}.$$

Let's analyze the commutator. All its terms contain at least one derivative hitting $g^{\mu\nu}$ so it will be a combination of the terms of the form

$$\partial^\beta \partial_j g_\sigma^{\mu\nu}\partial^\gamma\partial_\mu\partial_\nu u$$

with $|\beta| + |\gamma| = |\alpha| - 1$. Since $g_\sigma^{00} = -1$, we can assume that at most one of the indices μ, ν is zero. This is used below.

Set $g_0 = g(t, x, 0, 0)$ and write

$$\partial^\beta \partial_j g_v^{\mu\nu} \partial^\gamma \partial_\mu \partial_\nu u = \partial^\beta \partial_j (g_v^{\mu\nu} - g_0^{\mu\nu}) \partial^\gamma \partial_\mu \partial_\nu u + \partial^\beta \partial_j g_0^{\mu\nu} \partial^\gamma \partial_\mu \partial_\nu u$$

We have,

$$\| \partial^\beta \partial_j g_0^{\mu\nu} \partial^\gamma \partial_\mu \partial_\nu u \|_0 \leq C_I (\| \partial^\gamma \bar{\partial}^2 u \|_0 + \| \partial^\gamma \bar{\partial} \partial_t u \|_0)$$

$$\leq C_I' (\| u \|_{s+1} + \| \partial_t u \|_s) \leq C_I' \mathcal{M}_s(u)$$

For the second term, we first consider the inequality

$$\| \partial^\alpha u \, \partial^\beta v \|_0 \leq C (\| u \|_s \| v \|_{L^\infty} + \| u \|_{L^\infty} \| v \|_s), \quad |\alpha| + |\beta| = s$$

to get

$$\| \partial^\beta \partial_i (g_v^{\mu\nu} - g_0^{\mu\nu}) \partial^\gamma \partial_\mu \partial_\nu u \|_0 \leq C (\| \partial_i (g_v^{\mu\nu} - g_0^{\mu\nu}) \|_{s-1} \| \partial_\mu \partial_\nu u \|_{L^\infty}$$

$$+ \| \partial_i (g_v^{\mu\nu} - g_0^{\mu\nu}) \|_{L^\infty} \| \partial_\mu \partial_\nu u \|_{s-1}$$

Because μ, ν contain at most one index 0, we have

$$\| \partial_\mu \partial_\nu u \|_{L^\infty} \leq \mathcal{N}(u), \quad \| \partial_\mu \partial_\nu u \|_{s-1} \leq \mathcal{M}_s(u).$$

Also, $\| \partial_i g_0^{\mu\nu} \|_{L^\infty} \leq C_I$ and

$$\| \partial_i g_v^{\mu\nu} \|_{L^\infty} \leq \| (\partial_i g_v^{\mu\nu}) \partial_i (v, \partial v) \|_{L^\infty} \leq J_{\alpha, I}(|v|, |\partial v|) \| \bar{\partial} \partial v \|_{L^\infty}$$

$$\leq F_I (\mathcal{N}(v))$$

Thus

$$\| \partial^\beta \partial_i (g^{\mu\nu}_v - g^{\mu\nu}_0) \partial^\gamma \partial_\mu \partial_\nu u \|_0 \leq C \sum_{\mu,\nu} \| \partial_i (g^{\mu\nu}_v - g^{\mu\nu}_0) \|_{s-1} \mathcal{N}(u)$$

$$+ F_I(\mathcal{N}(v)) \mathcal{M}_s(u).$$

We have to estimate $\| \partial_i (g^{\mu\nu}_v - g^{\mu\nu}_0) \|_{s-1} \leq \| g^{\mu\nu}_v - g^{\mu\nu}_0 \|_s$.
We consider, for $|\alpha| \leq s$:

$\partial^\alpha (g^{\mu\nu}_v - g^{\mu\nu}_0) = \partial^\alpha \int_0^1 g^{\mu\nu}(t, x, \tau v, \tau \partial v) d\tau$. The chain rule gives, symbolically, $\partial^\alpha (g^{\mu\nu}(t, x, \tau v, \tau \partial v)) = \partial^\rho g^{\mu\nu}(t,x,\tau,v) \partial^\beta v \, \partial^\delta \partial v$, $|\rho| + |\beta| + |\delta| = |\alpha|$, so, using our assumptions,

$| \partial^\alpha (g^{\mu\nu}(t, x, \tau v, \tau \partial v))| \leq J_{I,\alpha}(|v|, |\partial v|) \, | \partial^\alpha \partial v| \leq F_I(\mathcal{N}(v)) \, |\partial^\alpha \partial v|$, thus

$$\| \partial^\alpha (g^{\mu\nu}_v - g^{\mu\nu}_0) \|_0 \leq F_I(\mathcal{N}(v)) \| \partial^\alpha \partial v \|_0 \leq F_I(\mathcal{N}(v)) (\|v\|_{s+1} + \|\partial_t v\|_s)$$

$$\leq F_I(\mathcal{N}(v)) \mathcal{M}_s(v).$$

We conclude that

$$\| [g^{\mu\nu}_v \partial_\mu \partial_\nu, \partial^\alpha] u \|_0 \leq F_I(\mathcal{N}(v)) \mathcal{M}_s(v) \mathcal{N}(u) + F_I(\mathcal{N}(v)) \mathcal{M}_s(u) + C_I \mathcal{M}_s(u)$$

$$\leq F_I(\mathcal{N}(v)) \left(\mathcal{N}(u) \mathcal{M}_s(v) + \mathcal{M}_s(u) \right)$$

Therefore,

$$\partial_t E_s \leq \|f_v\|_s E_s^{1/2} + F_I(\mathcal{N}(v)) E_s + F_I(\mathcal{N}(v)) (\mathcal{N}(u) \mathcal{M}_s(v) + \mathcal{M}_s(u)) E_s^{1/2}$$

Writing $f(t,x,v,\partial v) = f(t,x,v,\partial v) - f(t,x,0,0) + f(t,x,0,0)$

$$= \int_0^1 f(t,x,\tau v, \tau \partial v)\, d\tau + f(t,x,0,0)$$

and arguing as above in the estimate of $g_v - g_0$, we find

$\| f_v \|_s \leq C_I + F_I(\mathcal{W}(v)) \mathcal{M}_s(v)$. Hence

$$\partial_t E_s \leq F_I(\mathcal{W}(v)) E_s + \Big(C_I + F_I(\mathcal{W}(v)) \mathcal{M}_s(v)$$
$$+ F_I(\mathcal{W}(v)) \big(\mathcal{W}(u) \mathcal{M}_s(v) + \mathcal{M}_s(u) \big) \Big) E_s^{1/2}, \text{ or}$$

$$\boxed{\partial_t E_s \leq F_I(\mathcal{W}(v)) E_s + \Big[C_I + F_I(\mathcal{W}(v)) \big(\mathcal{M}_s(v) + \mathcal{W}(u) \mathcal{M}_s(v) + \mathcal{M}_s(u) \big) \Big] E_s^{1/2}}$$

Write $C_I E_s^{1/2} \leq \frac{1}{2} C_I^2 + \frac{1}{2} E_s$, $\mathcal{M}_s(v) E_s^{1/2} \leq \frac{1}{2} \mathcal{M}_s^2(v) + \frac{1}{2} E_s$

$\mathcal{W}(u) \mathcal{M}_s(v) E_s^{1/2} = \mathcal{W}^{1/2}(u) \mathcal{M}_s(v) \mathcal{W}^{1/2}(u) E_s^{1/2} \leq \mathcal{W}(u) \mathcal{M}_s^2(v) + \mathcal{W}(u) E_s$

$\leq F_I(\mathcal{W}(u)) (\mathcal{M}_s^2(v) + E_s)$, and $\mathcal{M}_s(u) E_s^{1/2} \leq C_I E_s^{1/2} E_s^{1/2} \leq C_I E_s$. We find:

$$\partial_t E_s \leq C_I + F_I(\mathcal{W}(v), \mathcal{W}(u)) \big(\mathcal{M}_s^2(v) + E_s \big),$$

which is the desired inequality for E_s. For the other inequality, divide the previous inequality for $\partial_t E_s$ by $E_s^{1/2}$, use $E_s^{1/2} \leq C_I \mathcal{M}_s(u)$, to find

$$\partial_t E_s^{1/2} \leq C_I + F_I(\mathcal{W}(v)) \big((1 + \mathcal{W}(u)) \mathcal{M}_s(v) + \mathcal{M}_s(u) \big)$$

which leads to the result.

Local well-posedness

We now prove local well-posedness for $g^{\mu\nu}(u,\partial u)\partial_\mu\partial_\nu u = f(u,\partial u)$. We will need to compare solutions, so we consider, for $i=1,2$:

$$
\begin{aligned}
g^{\mu\nu}_{v_i}\partial_\mu\partial_\nu u_i &= f_{v_i}, \\
u_i(0) &= u_{0,i}, \\
\partial_t u_i(0) &= u_{1,i}.
\end{aligned}
$$

We'll show that for $t \in [0,T]$

$$
\mathcal{M}_0(u) \le C'_I\, e^{\int_0^t F_I(\mathcal{N}(v_2))} \left(\mathcal{M}_0(u)(0) + \int_0^t F_I(\mathcal{N}(v_1), \mathcal{N}(v_2), \mathcal{N}(u_1))\, \mathcal{M}_0(v) \right)
$$

where $u = u_2 - u_1$, $v = v_2 - v_1$.

We see that u satisfies

$$
\begin{aligned}
g^{\mu\nu}_{v_2}\partial_\mu\partial_\nu u &= g^{\mu\nu}_{v_2}\partial_\mu\partial_\nu u_2 - g^{\mu\nu}_{v_2}\partial_\mu\partial_\nu u_1 \\
&= g^{\mu\nu}_{v_2}\partial_\mu\partial_\nu u_2 - g^{\mu\nu}_{v_1}\partial_\mu\partial_\nu u_1 + (g^{\mu\nu}_{v_1} - g^{\mu\nu}_{v_2})\partial_\mu\partial_\nu u_1 \\
&= f_{v_2} - f_{v_1} + (g^{\mu\nu}_{v_1} - g^{\mu\nu}_{v_2})\partial_\mu\partial_\nu u_1
\end{aligned}
$$

Set

$$
E = \frac{1}{2}\int_{\mathbb{R}^n} \left(-g^{00}_{v_2}|\partial_t u|^2 + g^{ij}_{v_2}\partial_i u \cdot \partial_j u + |u|^2 \right)
$$

Proceeding as in the zeroth order energy estimate:

$$\partial_t E \le F_I(W(v_2))E + \|f_{v_2} - f_{v_1}\|_0 E^{1/2} + \|(g^{\mu\nu}_{v_1} - g^{\mu\nu}_{v_2})\partial_\mu \partial_\nu u_1\|_0 E^{1/2}.$$

(Recall that in the referred energy estimate the term in power one in the energy was multiplied by $F_I(W(v))$ which came from differentiating the coefficients g_v. Here these coefficients are g_{v_2}.)

Since $g^{00}_{v_1} = g^{00}_{v_2} = -1$, at most one index in the last term is zero, thus

$$\|(g^{\mu\nu}_{v_1} - g^{\mu\nu}_{v_2})\partial_\mu \partial_\nu u_1\|_0 \le C\, W(u_1)\, \|g_{v_1} - g_{v_2}\|_0 .$$ Write

$$g_{v_1} - g_{v_2} = \int_0^1 \partial_\tau g(x, t, \tau v_1 + (1-\tau) v_2, \tau \partial v_1 + (1-\tau)\partial v_2)\, d\tau$$

$$= \int_0^1 \partial g(x, t, \tau v_1 + (1-\tau) v_2, \tau \partial v_1 + (1-\tau)\partial v_2)\cdot(v_1 - v_2, \partial v_1 - \partial v_2)$$

so

$$\|g_{v_1} - g_{v_2}\|_0 \le J_{d,I}(|v_1|, |v_2|, |\partial v_1|, |\partial v_2|)(\|v_1 - v_2\|_0 + \|\partial v_1 - \partial v_2\|_0)$$

$$\le F_I(\mathcal{N}(v_1), W(v_2))\, \mathcal{M}_0(v).$$

Estimating $f_{v_2} - f_{v_1}$ similarly and using Gronwall gives the result.

We are now ready to start the proof of local well-posedness.

Approximating sequence

Let $u_{0,\ell}$, $u_{1,\ell}$ be a sequence in C_0^∞ converging in $H^{s+1}\times H^s$ to u_0, u_1. Set $K_0 = \|u_0\|_{s+1} + \|u_1\|_s$. We can assume that $\|u_{0,\ell}\|_{s+1} + \|u_{1,\ell}\|_s \leq K_0 + 1$. Set $v_0 = u_{0,0}$. Define inductively $g_{\ell+1} = g_{v_\ell}$, $f_{\ell+1} = f_{v_\ell}$, and $v_{\ell+1}$ by solving

$$g^{\mu\nu}_{\ell+1}\partial_\mu\partial_\nu v_{\ell+1} = f_{\ell+1},$$

$$v_{\ell+1}(0) = u_{0,\ell+1},$$

$$\partial_t v_{\ell+1}(0) = u_{1,\ell+1}.$$

Then $v_{\ell+1}$ is smooth and by finite propagation speed it has spatial compact support (possibly depending on T) for $t\in[0,T]$.

Boundedness

Assume inductively that

$$\mathcal{M}_s(v_{\ell-1}) \leq \mathcal{C} \quad \text{and} \quad \mathcal{M}_s(v_{\ell-2}) \leq \mathcal{C}$$

for some $\mathcal{C}$, and $0\leq t\leq T$.

Using Sobolev's embedding theo, we have

$$\mathcal{W}(v_\ell) = \sum_{|\alpha|+j\leq 2} \sup_{x\in\mathbb{R}^n} |\partial^\alpha \partial_t^j v_\ell| \leq \|\partial^2 v_\ell\|_{L^\infty} + \|\partial\partial_t v_\ell\|_{L^\infty} + \|\partial_t^2 v_\ell\|_{L^\infty}$$

$$\leq C'\|v_\ell\|_{s+1} + C'\|\partial_t v_\ell\|_s + \|\partial_t^2 v_\ell\|_{L^\infty}$$

$$\leq C' \mathcal{M}_s(v_\ell) + \|\partial_t^2 v_\ell\|_{L^\infty}.$$

From the equation:

$$\|\partial_t^2 v_\ell\|_{L^\infty} \leq C'\|g^{-1}_{v_{\ell-1}}\|_{L^\infty}\left(\|\partial\partial_t v_\ell\|_{L^\infty} + \|\partial^2 v_\ell\|_{L^\infty} + \|f_{v_{\ell-1}}\|_{L^\infty}\right)$$

$$\leq C'\|g^{-1}_{v_{\ell-1}}\|_{L^\infty}\left(\mathcal{M}_s(v_\ell) + \|f_{v_{\ell-1}}\|_{L^\infty}\right),$$ where we used Sobolev embedding again. Now, from our assumptions:

$$\|g^{-1}_{v_{\ell-1}}\|_{L^\infty} \leq J_{\alpha,\beta}(\|v_{\ell-1}\|_{L^\infty}, \|\partial v_{\ell-1}\|_{L^\infty})$$

$$\leq F_I(\|v_{\ell-1}\|_{s+1}, \|\partial_t v_{\ell-1}\|_s) \leq F_I(\mathcal{M}_s(v_{\ell-1}))$$

$$\leq F_I(\mathcal{E}),$$ using the induction hypothesis in the last step. $\|f_{v_{\ell-1}}\|_{L^\infty}$ is similarly estimated. Then

$$\mathcal{W}(v_\ell) \leq F_I(\mathcal{E})(1+\mathcal{M}_s(v_\ell)).$$

Note that applying this inequality to $\ell \mapsto \ell-1$, using the induction hypothesis (that also holds for $\ell-2$, so we still get $F_I(\mathcal{E})$) we have

$$\mathcal{W}(v_{\ell-1}) \leq F_I(\mathcal{E}).$$

We now use the above into the energy estimate (with $u \mapsto v_\ell$, $v \mapsto v_{\ell-1}$):

$$\mathcal{M}_s(v_\ell) \leq C_I \mathcal{M}_s(v_\ell)(0) + \int_0^t \Big(C_I + F_I(\underbrace{\mathcal{W}(v_{\ell-1})}_{\leq F_I(\mathcal{C})})\Big)\Big(\big(1+\overbrace{\mathcal{W}(v_{\ell-1})}^{\leq F_I(\mathcal{C})}\big)\underbrace{\mathcal{M}_s(v_{\ell-1})}_{\leq \mathcal{C}} + \mathcal{M}_s(v_\ell)\Big)$$

$$\leq C_I\Big(\mathcal{M}_s(v_\ell)(0) + F_I(\mathcal{C})\int_0^t (1+\mathcal{M}_s(v_\ell))\Big).$$

Gronwall gives:

$$\mathcal{M}_s(v_\ell) \leq C_I\big(\underbrace{\mathcal{M}_s(v_\ell)(0)}_{\leq K_0+1} + t F_I(\mathcal{C})\big)e^{tF_I(\mathcal{C})}.$$

Choosing T small enough (depending on K_0, C_I, and the structure of F_I, thus ultimately depending only on K_0 and I) we have that $\mathcal{M}_s(v_\ell) \leq \mathcal{C}$, $0 \leq t \leq T$, for all ℓ.

To conclude the induction, we need to verify it for $\ell=1,2$, i.e., we need $\mathcal{M}_s(v_0) \leq \mathcal{C}$, $\mathcal{M}_s(v_1) \leq \mathcal{C}$. For $\ell=1$, i.e., v_0, we have $\mathcal{M}_s(v_0) = \mathcal{M}_s(u_{0,0}) \leq K_0+1$, so we choose $\mathcal{C} > K_0+1$.

For $\ell=2$, i.e., v_1, we apply the above inductive argument.

For this, we need $\mathcal{N}(v_0) \leq F_I(\mathcal{E})$, which above we obtained using the hypothesis on $l-2$, which here would correspond to v_{-1} (see above where we considered $l \mapsto l-1$). However, we get $\mathcal{N}(v_0) \leq F_I(\mathcal{E})$ directly from the fact that v_0 is constant in time and from Sobolev embedding.

Lower norm convergence

We first observe that that the solutions v_l are in $C^0([0,T], H^{s+1}) \cap C^1([0,T], H^s)$. This follows from the linear theory. Alternatively, recall that our solutions have the property that for any compact $[T_1, T_2]$, there is a compact $K \subseteq \mathbb{R}^n$ such that $v(t,x) = 0$ for $T_1 \leq t \leq T_2$, $x \notin K$. Smooth functions with this property belong to $C^l(\mathbb{R}, H^s)$ for any l, s.

We'll show that $\{v_l\}$ converges in $C^0([0,T], H^1) \cap C^1([0,T], L^2)$. Apply the estimate for difference of solutions with $v_2 = v_l$, $v_1 = v_{l-1}$, $u_2 = v_{l+1}$, $u_1 = v_l$:

$$\mathcal{M}_0(v_{l+1} - v_l) \leq C_I' e^{\int_0^t F_I(\mathcal{N}(v_l))} \Big(\mathcal{M}_0(v_{l+1} - v_l)(0)$$

$$+ \int_0^t F_I(\mathcal{N}(v_{l-1}), \mathcal{N}(v_l)) \mathcal{M}_0(v_l - v_{l-1}) \Big)$$

Above we showed that $W(v_\ell) \leq F_I(\mathcal{E})(1 + M_s(v_\ell))$, which implies $W(v_\ell) \leq F_I(\mathcal{E})$ for all ℓ in view of the boundedness of the sequence. Thus

$$M_0(v_{\ell+1} - v_\ell) \leq C_I' e^{t F_I(\mathcal{E})} \left(M_0(v_{\ell+1} - v_\ell)(0) + F_I(\mathcal{E}) \int_0^t M_0(v_\ell - v_{\ell-1}) \right).$$

Put $a_\ell = \sup_{0 \leq t \leq T} M_0(v_{\ell+1} - v_\ell)(t)$. We can choose T small enough and the sequence approximating the initial data such that $t F_I(\mathcal{E}) \leq \frac{1}{2}$ and $C_I' e^{t F_I(\mathcal{E})} M_0(v_{\ell+1} - v_\ell)(0) \leq 2^{-\ell}$, thus $a_\ell \leq 2^{-\ell} + \frac{1}{2} a_{\ell-1}$. Then $a_2 \leq \frac{1}{4} + \frac{1}{2} a_1$, $a_3 \leq \frac{1}{8} + \frac{1}{2} a_2 \leq \frac{1}{8} + \frac{1}{8} + \frac{1}{4} a_1, \ldots, a_\ell \leq \frac{\ell - 1}{2^\ell} + \frac{a_1}{2^{\ell-1}}$.

Then $M_0(v_{\ell+j} - v_\ell) \leq M_0(v_{\ell+j} - v_{\ell+j-1}) + \cdots + M_0(v_{\ell+1} - v_\ell)$

$$\leq \frac{\ell + j - 2}{2^{\ell+j-1}} + \frac{a_1}{2^{\ell+j-2}} + \cdots + \frac{\ell - 1}{2^\ell} + \frac{a_1}{2^{\ell-1}}$$

which can be made arbitrarily small upon taking ℓ, j sufficiently large since the series $\sum_1^\infty \frac{\ell}{2^\ell}$ converges (hence its tail can be made small). We conclude that $\{v_\ell\}$ is Cauchy in $C^0([0,T], H^1) \cap C^1([0,T], L^2)$, hence it has a limit.

Higher norm convergence

Using interpolation we have (with $0<a,b,c,d<1$, $a+b=1=c+d$)

$\| v_{\ell+j} - v_\ell \|_{r+1} + \| \partial_t v_{\ell+j} - \partial_t v_\ell \|_r \lesssim \| v_{\ell+j} - v_\ell \|_{s+1}^a \| v_{\ell+j} - v_\ell \|_1^b$
$+ \| \partial_t v_{\ell+j} - \partial_t v_\ell \|_s^c \| \partial_t v_{\ell+j} - \partial_t v_\ell \|_0^d \lesssim (2C)^a \| v_{\ell+j} - v_\ell \|_1^b$
$+ (2C)^c \| \partial_t v_{\ell+j} - \partial_t v_\ell \|_0^d$, where we used the boundedness of the sequence and $0<a,b,c,d<1$, $0<r<s$. We conclude that $\{v_\ell\}$ is Cauchy in $C^0([0,T], H^{r+1}) \cap C^1([0,T], H^r)$, $0 \leq r < s$. Since $s > \frac{n}{2}+1$, we can take $r > \frac{n}{2}+1$, and then Sobolev embedding gives that $\{v_\ell\}$ converges in $C^0([0,T], C^2)$ and $\{\partial_t v_\ell\}$ converges in $C^1([0,T], C^1)$. From the equation, we have that $\partial_t^2 v_\ell$ can be written in terms of $\partial^2 v_\ell$, $\partial \partial_t v_\ell$, and $\partial v_{\ell-1}$, hence $\{\partial_t^2 v_\ell\}$ converges in $C^0([0,T], C^0)$. We conclude that $\sum_{|\alpha|+j \leq 2} \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^n} | \partial^\alpha \partial_t^j (v_{\ell+j}(t,x) - v_\ell(t,x)) | \longrightarrow 0$ as $\ell \to \infty$, i.e., the sequence converges in $C^2([0,T] \times \mathbb{R}^n, \mathbb{R}^N)$. It follows that the limit u satisfies the equation.

We next have to show that $u \in C^0([0,T], H^{s+1}) \cap C^1([0,T], H^s)$. From the above we already know it belongs to this space with s replaced by $r < s$ and and converges with s replaced by r. We now recall that if a sequence is bounded in H^s and converges in H^r, then the limit is also in H^s and satisfies the same bound (for, boundedness in H^s gives a weakly convergent subsequence $v_{\ell_k} \rightharpoonup v_\infty \in H^s$, and also $\|v_\infty\|_s \leq \liminf \|v_{\ell_k}\|_s \leq C(\mathcal{E})$ by weak lower semi-continuity of the norm, where $C(\mathcal{E})$ comes from the interpolation bound above; by embedding, $v_{\ell_k} \to v_\infty$ in H^r, but $v_\ell \to u$ in H^r so $u = v_\infty$). Thus:

$$\|u\|_{s+1} + \|\partial_t u\|_s \leq C(\mathcal{E}) .$$

The above argument is pointwise in time, i.e., $v_{\ell_k}(t) \rightharpoonup u(t)$ for each fixed t.

This shows that, for each fixed t, $u(t) \in H^{s+1}$ and $\partial_t u(t) \in H^s$. It does not yet show the continuity and differentiability in t, which will be showed below. Before proceeding, we give an alternative proof that $u(t) \in H^{s+1}$, $\partial_t u(t) \in H^s$. We have

$$\|u\|_{r+1} + \|\partial_t u\|_r = \lim_{\ell \to \infty} \left(\|v_\ell\|_{r+1} + \|\partial_t v_\ell\|_{r+1} \right) \leq C(\mathcal{E})$$

Using the Fourier transform definition of H^r we have

$$\|u\|^2_{r+1} = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 (1 + |\xi|^2)^{r+1} .$$

Set $w_r(\xi) = |\hat{u}(\xi)|^2 (1+|\xi|^2)^{r+1}$. This converges pointwise to $w_s(\xi)$ as $r \to s$, the family w_r is increasing in r, and their L^1-norm is bounded by $C'(t)$, thus independently of r. Thus, by the monotone convergence theorem, $w_s \in L^1$, which means $u \in H^{s+1}$ and satisfies the same bound. Similarly for $\partial_t u \in H^s$.

<u>Continuity: weak</u>

We first show that u is weakly continuous in t. I.e., we will show that for any $\varphi \in H^{-s-1}$ the function $t \mapsto \varphi(u(t))$ is continuous. Let $\varphi \in H^{-s-1}$. Then

$$\langle \varphi, w \rangle = \int_{\mathbb{R}^n} \varphi\, w$$

for all $w \in H^{s+1}$.

Notice also the abuse of notation: precisely, given a φ in the dual of H^{s+1}, it is represented by a $f_\varphi \in H^{-s-1}$ such that $\langle \varphi, w \rangle = \int_{\mathbb{R}^n} f_\varphi\, w$.

Let $\{\varphi_k\}$ be a sequence of Schwartz functions converging in H^{-s-1} to φ. We have:

$$\langle \varphi, u(t)\rangle - \langle \varphi, v_\ell(t)\rangle = \langle \varphi, u(t)\rangle - \langle \varphi_j, u(t)\rangle + \langle \varphi_j, u(t)\rangle - \langle \varphi, v_\ell(t)\rangle$$

$$= \langle \varphi, u(t)\rangle - \langle \varphi_j, u(t)\rangle + \langle \varphi_j, u(t)\rangle - \langle \varphi_j, v_\ell(t)\rangle + \langle \varphi_j, v_\ell(t)\rangle - \langle \varphi, v_\ell(t)\rangle.$$

$$|\langle \varphi, u(t)\rangle - \langle \varphi, v_\ell(t)\rangle| \leq |\langle \varphi - \varphi_j, u(t)\rangle| + |\langle \varphi - \varphi_j, v_\ell(t)\rangle|$$

$$+ |\langle \varphi_j, u(t) - v_\ell(t)\rangle| \leq \|\varphi - \varphi_j\|_{-s-1}\left(\|u(t)\|_{s+1} + \|v_\ell(t)\|_{s+1}\right) + |\langle \varphi_j, u(t) - v_\ell(t)\rangle|$$

$$\leq \|\varphi - \varphi_j\|_{-s-1} C'(T) + |\langle \varphi_j, u(t) - v_\ell(t)\rangle|.$$

Choose j_0 large enough so that the first term is $< \frac{\varepsilon}{2}$ for all $j \geq j_0$. For the second term, we note that since φ_{j_0} is a Schwartz function, it defines a bounded linear functional on H^{r+1}, $r < s$, so that

$$|\langle \varphi_{j_0}, u(t) - v_\ell(t)\rangle| \leq \|\varphi_{j_0}\|_{-r-1} \|u(t) - v_\ell(t)\|_{r+1} \leq \|\varphi_{j_0}\|_{-r} \sup_{0 \leq t \leq T} \|u(t) - v_\ell(t)\|_{r+1}.$$

By the above convergence in $C^0([0,T], H^{r+1})$, we can choose l_0, depending on φ_{j_0}, such that this term is $< \frac{\varepsilon}{2}$ for $l \geq l_0$. Thus

$$|\langle \varphi, u(t)\rangle - \langle \varphi, v_\ell(t)\rangle| < \varepsilon \quad \text{for all } l \geq l_0 \text{ and all } t \in [0,T].$$

Thus, v_ℓ converges uniformly (in t) to u in the weak topology here employed. Since $v_\ell \in C^0([0,T], H^s)$, they are weakly continuous and so is u because it is the uniformly limit of continuous functions in the weak topology. The proof for $\partial_t u$ is similar

Continuity: strong

We now prove strong continuity. We will first show that u and $\partial_t u$ are right continuous at $t=0$ in the desired topology, i.e.,

$$\lim_{t\to 0^+} \left(\| u(t) - u(0) \|_{s+1} + \| \partial_t u(t) - \partial_t u(0) \|_s \right) = 0.$$

For this, we will need the following energy estimate:

$$\bar{E}_s(t) \lesssim \left(\bar{E}_s(0) + t\, C_I \right) e^{\int_0^t F_I(N(u))},$$ where

$\bar{E}_s(t) = \bar{E}_s(u) = E_s(u,u)(t)$, which we will prove later.

Set $h^{ij}(x) = g^{ij}(0,x,u,\partial u)$ and define an inner product on $H^{s+1}\times H^s$ by

$$((v_1,v_2),(w_1,w_2)) = \frac{1}{2}\sum_{|\alpha|\le s}\int_{\mathbb{R}^n}\left(h^{ij}\partial_i\partial^\alpha v_1 \cdot \partial_j \partial^\alpha w_1 + \partial^\alpha v_1 \cdot \partial^\alpha w_1 + \partial^\alpha v_2 \cdot \partial^\alpha w_2 \right)$$

which gives an equivalent inner product in light of our assumptions and results established so far. We have:

$$((u-u_0, \partial_t u - u_1),(u-u_0,\partial_t u - u_1)) = ((u,\partial_t u)-(u_0,u_1),(u,\partial_t u)-(u_0,u_1))$$

$$= ((u,\partial_t u),(u,\partial_t u)) + ((u_0,u_1),(u_0,u_1)) - 2((u,\partial_t u),(u_0,u_1))$$

For convenience, we write explicitly the expression for $\bar{E}_s$:

$$\bar{E}_s = \frac{1}{2} \sum_{|\alpha| \leq s} \int_{\mathbb{R}^n} \left(-g_u^{00} |\partial_t \partial^\alpha u|^2 + g_u^{ij} \partial_i \partial^\alpha u \cdot \partial_j \partial^\alpha u + |\partial^\alpha u|^2 \right)$$

Since $h = g(0, x, u, \partial u)$, we have

$$\bar{E}_s(0) = \frac{1}{2} \sum_{|\alpha| \leq s} \int_{\mathbb{R}^n} \left(|\partial_t \partial^\alpha u|^2 + h^{ij} \partial_i \partial^\alpha u \cdot \partial_j \partial^\alpha u + |\partial^\alpha u|^2 \right) \Big|_{t=0}$$

$$= \left((u_0, u_1), (u_0, u_1) \right).$$

Since the map $u \mapsto ((u, \partial_t u), (u_0, u_1))$ defines a linear functional, by the weak continuity established above, this term converges to $\bar{E}_s(0)$ as $t \to 0^+$. We obtain:

$$\limsup_{t \to 0^+} \left((u - u_0, \partial_t u - u_1), (u - u_0, \partial_t u - u_1) \right)$$

$$= \limsup_{t \to 0^+} \left((u, \partial_t u), (u, \partial_t u) \right) - \bar{E}_s(0).$$

$$= \limsup_{t \to 0^+} \left[((u, \partial_t u), (u, \partial_t u)) - \bar{E}_s(t) + \bar{E}_s(t) \right] - \bar{E}_s(0)$$

$$\leq \limsup_{t \to 0^+} \left[((u, \partial_t u), (u, \partial_t u)) - \bar{E}_s(t) \right] + \limsup_{t \to 0^+} \bar{E}_s(t) - \bar{E}_s(0).$$

The above energy estimate for $\bar{E}_s(t)$ gives

$$\limsup_{t \to 0^+} \bar{E}_s(t) \leq \bar{E}_s(0)$$

so that

$$\limsup_{t\to 0^+} \left((u-u_0, \partial_t u - u_1), (u-u_0, \partial_t u - u_1) \right)$$

$$\leq \limsup_{t\to 0^+} \left[\left((u, \partial_t u), (u, \partial_t u) \right) - \tilde{E}_s(t) \right].$$

Write $\left((u, \partial_t u), (u, \partial_t u) \right) - \tilde{E}_s(t) =$

$$\frac{1}{2} \sum_{|\alpha| \leq s} \int_{\mathbb{R}^n} \left(|\partial_t \partial^\alpha u|^2 + h^{ij} \partial_i \partial^\alpha u \cdot \partial_j \partial^\alpha u + |\partial^\alpha u|^2 \right)$$

$$- \frac{1}{2} \sum_{|\alpha| \leq s} \int_{\mathbb{R}^n} \left(-g_u^{00} |\partial_t \partial^\alpha u|^2 + g_u^{ij} \partial_i \partial^\alpha u \cdot \partial_j \partial^\alpha u + |\partial^\alpha u|^2 \right)$$

$$= \frac{1}{2} \sum_{|\alpha| \leq s} \int_{\mathbb{R}^n} \left(h^{ij} - g_u^{ij} \right) \partial_i \partial^\alpha u \cdot \partial_j \partial^\alpha u$$

$$\leq C \| h^{-1} - g_u^{-1} \|_{L^\infty} \| u \|_{s+1}^2 \leq \| h^{-1} - g_u^{-1} \|_{L^\infty} C(\tau).$$

To top order, $h^{-1} - g_u^{-1} = \partial u_0 - \partial u$, so $\| h^{-1} - g_u^{-1} \|_{L^\infty} \leq \| \partial u_0 - \partial u \|_{L^\infty}$ $\leq \| u_0 - u \|_{C^1} \leq C \| u_0 - u \|_s$, since $s > \frac{n}{2} + 1$. Because we have already proved continuity in H^{r+1} for $r < s$, this term goes to zero as $t \to 0^+$. Thus

$$\limsup_{t\to 0^+} \left((u-u_0, \partial_t u - u_1), (u-u_0, \partial_t u - u_1) \right) = 0.$$

The left hand side is precisely the limsup of the norm difference we want to estimate, and since it is always ≥ 0, we have showed right continuity at zero.

Next, we observe that the equations are time reversible, i.e., we can equally obtain existence on an interval $[-T, 0]$. Repeating the above we obtain left continuity at zero, hence continuity, at zero.

To prove continuity at a $t_0 \neq 0$, consider the solution u and put $\tilde{u}_0 = u(t_0)$, $\tilde{u}_1 = \partial_t u(t_0)$. We repeat the above arguments and solve the Cauchy problem with data $\tilde{u}_0, \tilde{u}_1$ at $t = t_0$, obtaining a solution $\tilde{u}$ which, by the above with t_0 instead of zero, is continuous at t_0. By uniqueness (to be proved below) u and $\tilde{u}$ coincide in a neighborhood of t_0, thus u is continuous at t_0.

Uniqueness

We apply the inequality

$$\mathcal{M}_0(u) \leq C_I' \, e^{\int_0^t F_I(\mathcal{W}(v_2))} \left(\mathcal{M}_0(u)(0) + \int_0^t F_I(\mathcal{W}(v_1), \mathcal{W}(v_2), \mathcal{W}(u_1)) \mathcal{M}_0(v) \right)$$

with $v = u$ and the bounds in terms of $\mathcal{C}$ and use Gronwall.

Energy estimate for u

We now establish the estimate

$$\overline{E}_s(t) \leq \left(\overline{E}_s(0) + t \, C_I \right) e^{\int_0^t F_I(\mathcal{W}(u))}$$

used above. Recall the inequality

$$\partial_t E_s(v,u) \le C_I + F_I(N(v), N(u))\left(\mathcal{M}_s^2(v) + E_s(v,u)\right).$$

Apply this inequality with $v \mapsto v_\ell$, $u \mapsto v_{\ell+1}$, $u_0 \mapsto u_{0,\ell}$, $u_1 \mapsto u_{1,\ell}$:

$$E_s(v_\ell, v_{\ell+1}) \le E_s(v_\ell, v_{\ell+1})(0) + \int_0^t \left(C_I' + F_I(N(v_\ell), N(v_{\ell+1}))\right)\left(\mathcal{M}_s^2(v_\ell) + E_s(v_\ell, v_{\ell+1})\right)$$

We once again recall the definition of E_s:

$$E_s(v,u) = \frac{1}{2}\sum_{|\alpha|\le s}\int_{\mathbb{R}^n}\left(-g_v^{00}|\partial^\alpha \partial_t u|^2 + g_v^{ij}\partial^\alpha\partial_i u\cdot\partial^\alpha\partial_j u + |\partial^\alpha u|^2\right)$$

and estimate

$$|E_s(v_\ell, v_{\ell+1})(t) - E_s(u, v_{\ell+1})(t)| \le \|g_{v_\ell}^{-1} - g_u^{-1}\|_{L^\infty} C'(\varepsilon) \to 0 \text{ as } \ell \to \infty$$

since v_ℓ converges to u in C^2. This convergence in C^2 also implies that $N(v_\ell) \to N(u)$. Moreover, since $E_s(u, v_{\ell+1})(0) = E_s(u(0), v_{\ell+1}(0))$ converges to $E_s(u,u)(0)$ due to the construction of initial data $u_{0,\ell}$, $u_{1,\ell}$ for v_ℓ, we have $E_s(v_\ell, v_{\ell+1})(0) \to E_s(u,u)(0) = E_s(u)(0)$. Recall that we also have the bound comparing $\mathcal{M}_s$ and E_s, so that $\mathcal{M}_s^2(v_\ell) \le C_I' E_s(u, v_\ell)$.

Finally,

$$|\bar{E}_s(u,v_\ell)| \leq F_I(W(u))\, M_s(v_\ell) \leq C'(G),$$

which shows that $E_s(u,v_\ell)(t)$ is uniformly bounded in t on $[0,T]$. Hence, we can apply the reverse Fatou's lemma and, using the above remarks,

$$\limsup_{\ell\to\infty} E_s(u,v_\ell) \leq \bar{E}_s(u)(0) + \int_0^t \limsup_{\ell\to\infty} \Big(C_I'$$

$$+ F_I(W(v_\ell), W(v_{\ell+1}))\big(M_s^2(v_\ell) + E_s(v_\ell, v_{\ell+1})\big)$$

$$\leq \bar{E}_s(u)(0) + \int_0^t \Big(C_I' + F_I(W(u)) \limsup_{\ell\to\infty} E_s(u,v_\ell)\Big).$$

Gronwall now implies that

$$\limsup_{\ell\to\infty} E_s(u,v_\ell) \leq \left(\bar{E}_s(u)(0) + t\, C_I'\right) e^{\int_0^t F_I(W(u))}.$$

From the weak convergence previously established:

$$\lim_{\ell\to\infty} \frac{1}{2}\sum_{|\alpha|\leq s} \int_{\mathbb{T}^n} \left(-g_u^{00}\, \partial_t \partial^\alpha v_\ell \cdot \partial_t \partial^\alpha u + g_u^{ij}\, \partial_i \partial^\alpha v_\ell \cdot \partial_j \partial^\alpha u + \partial^\alpha v_\ell \cdot \partial^\alpha u \right)$$

$= \bar{E}_s(u)$. The result will now follow if we show that the left-hand side is bounded by

$$\bar{E}_s^{1/2}(u) \limsup_{\ell\to\infty} E_s^{1/2}(u,v_\ell)$$

since this will imply $\bar{E}_s^{1/2}(u) \leq \limsup_{\ell\to\infty} E_s^{1/2}(u,v_\ell)$.

Similarly to what was done before, the expression

$$\frac{1}{2}\sum_{|\alpha|\le s}\int_{\mathbb{R}^n}\left(g_u^{ij}\partial_i\partial^\alpha w_1\cdot\partial^\alpha w_2+\partial^\alpha w_1\cdot\partial^\alpha w_2+\partial^\alpha w_1\cdot\partial^\alpha w_2\right)$$

defines an inner product $(\cdot,\cdot)_u$ on $H^{s+1}\times H^s$, so in particular Cauchy-Schwarz holds. Thus

$$\frac{1}{2}\sum_{|\alpha|\le s}\int_{\mathbb{R}^n}\left(\partial_t\partial^\alpha u\cdot\partial_t\partial^\alpha v_\ell+g_u^{ij}\partial_i\partial^\alpha u\cdot\partial_j\partial^\alpha v_\ell+\partial^\alpha u\cdot\partial^\alpha v_\ell\right)=((u,\partial_t u),(v_\ell,\partial_t v_\ell))$$

$$\le \|(u,\partial_t u)\|_u\,\|(v_\ell,\partial_t v_\ell)\|_u=\bar{E}_s^{1/2}(u)\,\bar{E}_s^{1/2}(u,v_\ell),$$

as desired.

Remarks, comments

1. We defined $\mathcal{W}(u)$ involving up to two derivatives of u because we assumed $g = g(u, \partial u)$, so that when we need to estimate ∂g in L^∞ we obtain $\partial g \sim \partial^2 u$ in L^∞. If $g = g(u)$, as is the case for example in Einstein's equations, then we have $\partial g \sim \partial u$ and we have to control only the L^∞ norm of ∂u. Thus, in this case we can define $\mathcal{W}(u)$ involving only derivatives up to first order, and the above argument still goes through.

2. The proof of the bound for $\bar{E}_s$ cannot be obtained by a direct energy estimate on the solution u. This is because this would involve using Gronwall, which requires continuity in time. But we use the the bound on $\bar{E}_s$ to prove (strong) continuity in time.

3. The condition $s > \frac{n}{2}+1$ was used for the Sobolev embedding theorem.

4. We used Sobolev's embedding in two ways. First, to bound $\| \partial^2 v_\ell \|_{L^\infty} \leq C \| \partial^2 v_\ell \|_{s-1} \sim \| v_\ell \|_{s+1}, \| \partial_t v_\ell \|_s$ (using the equation for ∂_t^2). These terms come from $\mathcal{W}(v_\ell)$.

Second, to obtain that the convergence in H^{r+1}, $r<s$, gives convergence in C^2 $\left(r+1 > \frac{n}{2}+2\right)$. The C^2 convergence, in turn, was used to obtain $\|g^{-1}_{v_\ell} - g^{-1}_{u}\|_{L^\infty}C^1(\mathcal{E}) \to 0$ and $\mathcal{N}(v_\ell) \to \mathcal{N}(u)$.

Suppose now that $g = g(u)$, i.e., the metric does not depend on ∂u. We claim that in this case it suffices to assume $s > \frac{n}{2}$. To see this, first note that in this case we can take $\mathcal{N}(u) \sim \partial u$, as pointed out above. Then the argument with Sobolev embedding gives

$$\|\mathcal{N}(v_\ell)\|_{L^\infty} \sim \|\partial v\|_{L^\infty} \leq C\|\partial v\|_s \sim \|v\|_{s+1} + \|\partial_t v\|_s,$$

as desired.

The convergence in H^{r+1} in this case gives convergence in C^1 $\left(r+1 > \frac{n}{2}+1\right)$. This suffices because now

$$\|g^{-1}_u - g^{-1}_{v_\ell}\|_{L^\infty} C^1(\mathcal{E}) \sim \|\partial u - \partial v_\ell\|_{L^\infty} C^1(\mathcal{E}) \to 0 \text{ as } \ell \to \infty$$

and $\mathcal{N}(v_\ell) \to \mathcal{N}(u)$ since now $\mathcal{N}(u) \sim \partial u$.

We don't necessarily obtain a C^2 solution in this case though because the convergence is only in C^1. But we still do get a classical solution. For, we do have convergence in H^{r+1}. Since

$r+1 > \frac{n}{2}+1 \geq 2$ for $n \geq 2$, we have that $\partial^\alpha v_\ell$ converges pointwise almost everywhere to $\partial^\alpha u$ (possibly after passing to a subsequence).