Reviews in Mathematical Physics
Vol. 27, No. 6 (2015) 1550014 (45 pages)

DOI: 10.1142/S0129055X15500142

Remarks on the Einstein–Euler-Entropy system

Marcelo M. Disconzi

Department of Mathematics,
Vanderbilt University, Nashville,
Tennessee, 37240, USA
marcelo.disconzi@vanderbilt.edu

Received 29 July 2013
Accepted 21 June 2015
Published 15 July 2015

We prove short-time existence for the Einstein–Euler-Entropy system for non-isentropic fluids with data in uniformly local Sobolev spaces. The cases of compact as well as non-compact Cauchy surfaces are covered. The method employed uses a Lagrangian description of the fluid flow which is based on techniques developed by Friedrich, hence providing a completely different proof of earlier results of Choquet-Bruhat and Lichnerowicz. This new proof is specially suited for applications to self-gravitating fluid bodies. Along the way, we review some basic definitions and ideas, giving thus a relatively self-contained exposition that also serves as an introduction to many aspects of the problem.

Keywords: Einstein–Euler; relativistic fluids; first-order symmetric hyperbolic.

Mathematics Subject Classification 2010: 83C05, 35L03, 76B03

Contents

1. Introduction

The Einstein equations have been a source of several interesting problems in Physics, Analysis and Geometry. Despite the great deal of work that has been devoted to them, with many success stories, several important questions remain (see [13, 17, 52] for an account of what is currently known and some directions of future research). One of them is finding a satisfactory theory of isolated systems, such as stars, both from a perspective of the time development of the space-time, as well as from the point of view of the geometry induced on a space-like three-surface. To quote Rendall, "of the physical situations which can be described by the general theory of relativity, those which are at the present most accessible to observation are the isolated systems. In fact, all existing tests of Einstein's equations concern such situations. It is therefore important to have a theory of these systems which is as complete as possible, not only in terms of the range of phenomena which are covered but also with respect to logical and mathematical solidity" [47].

Stars are the prototypes of isolated systems. They are typically modeled by considering a region of space filled with a fluid and separated from an exterior that corresponds to vacuum. The properties of the fluid, such as perfection, viscosity, charge, etc., depend on the particular situation one is interested in. The dynamics of the fluid region is then described by Einstein equations coupled to matter, whereas vacuum Einstein equations hold on the complement of this set. From the point of view of the Cauchy problem, which will be the main case of interest in this work, there are two primary questions to be addressed.

(i) First, the solvability of Einstein equations in the two different situations of interest, i.e. coupled to fluid sources and in vacuum, should be addressed. Since the short-time existence for vacuum Einstein equations is well understood (see, e.g., [32]), this leaves us with the coupling to matter. A specific matter model has then to be chosen, and one of the most customary choices, sufficient for many applications, is that of a perfect fluid [2], in which case Einstein equations are coupled to the (relativistic) Euler equations. The well-posedness[a] of this system was proven by Choquet-Bruhat [10] and extended by Lichnerowicz [36] to include entropy — in which case the resulting system will be called the Einstein–Euler-Entropy system, whose equations are stated in Sec. 2.2.

(ii) The second question is more delicate and consists of trying to bring together the two different scenarios described in the previous paragraph, namely, vacuum and coupling to matter. More precisely, we attempt to formulate and solve the

[a] "Well-posedness" should be understood here in the context of General Relativity, where uniqueness is meant in a geometric sense, i.e. up to isometries.

Cauchy problem with an interior region — thought of as the star — governed by the Einstein–Euler (or Einstein–Euler-Entropy) equations and an exterior one that evolves according to the vacuum Einstein equations. This corresponds to a genuine free-boundary problem in that the boundary of the star cannot be prescribed for time $t > 0$, being rather a dynamic quantity that has to be determined from the evolution equations. The star boundary at time zero is obtained as the boundary of the support Ω of the initial matter density ϱ_0, which is a function on the initial Cauchy surface typically of the form

$$\varrho_0(x) = \begin{cases} f(x) > 0, & x \in \Omega, \\ 0, & x \notin \Omega. \end{cases} \tag{1.1}$$

Solving the Cauchy problem then requires a refined analysis of the boundary behavior of the quantities involved, where the change from $G_{\alpha\beta} = T_{\alpha\beta}$ to $G_{\alpha\beta} = 0$ causes severe technical difficulties (here $G_{\alpha\beta}$ and $T_{\alpha\beta}$ are the Einstein and stress-energy tensor, respectively).

Lindbom [40] has proven that any isolated static fluid stellar model ought to be spherically symmetric, generalizing a classical result of Carleman and Lichtenstein for Newtonian fluids [7, 38, 39]. More precisely, he has shown that a static asymptotically flat space-time that contains only a uniform density perfect fluid confined to a spatially compact world tube is necessarily spherical symmetric.

Under the assumption of spherical symmetry, it is possible to deal with many of the technicalities that arise, and a number of satisfactory and general results have been obtained. Rendall and Schmidt have proven existence and uniqueness results for global solutions of the Einstein–Euler system [49]. They also gave a detailed account of how the properties of these solutions depend on features of the equation of state. Kind and Ehlers have treated the mixed initial-boundary value problem in [35], where they have also given necessary and sufficient conditions for attaching the solutions they constructed to a Schwarzschild space-time. Makino [43] refined the results of Rendall and Schmidt by providing a general criterion for the equation of state, which ensures that the model has finite radius (and therefore finite mass). This result is used to study the linear stability of the equations of motion. Regarding the relation between mass and radius, in [33] the authors derive, among other results, interesting mass-radius theorems. A very extensive treatment of the Cauchy problem for spherically symmetric data is given in the work of Groah, Smoller and Temple [31]. These works do not exhaust all the results known in spherical symmetry; see the references in the above papers for more details.

If we drop the hypothesis of spherical symmetry, however, not much is known about the well-posedness of the Cauchy problem when the matter density is allowed to vanish outside a compact set, as in (1.1) — a situation generally referred to as a "fluid body". One could attempt to change ϱ_0 to a function that is always positive, but decays sufficiently fast and is very small in the region outside Ω. Unfortunately, this does not improve things considerably because generally the time-span of solutions cannot be shown to be uniform, and as a result, the domain of definition of the solution contains no space-like slice $t =$ constant (except obviously the $t = 0$ slice). Another problem which arises in this context is that the usual conformal method for solving the constraint equations [14, 37, 63] cannot be applied.

Following the ideas of Makino [42], who studied non-relativistic[b] gaseous stars, Rendall has shown that the above problems can be circumvented for fluids obeying certain equations of state [48] — although, as the author himself points out, the solutions obtained in these cases have the undesirable property of not including statically spherically symmetric space-times, and the restrictions on the equation of state are too strong. Initial data sets for the Einstein–Euler equations describing a fluid body have been successfully constructed by Brauer and Karp [3] and Dain and Nagy [16], but well-posedness of the system with such prescribed initial values has not yet been demonstrated. Therefore, the solution to the Cauchy problem for a self-gravitating isolated fluid body is still largely open (although some recent results of Brauer and Karp [4–6] address directly some of the technical issues we referred to).

An important step forward has been achieved by Friedrich in [23]. Using a Lagrangian description of the fluid motion, he has been able to derive a set of reduced equations which form a first-order symmetric hyperbolic system. While it has been known that the vacuum Einstein equations can be cast in such a form since the work of Fischer and Marsden [22], and the Einstein–Euler system had also been investigated within the formalism of first-order symmetric hyperbolic systems in the aforementioned work of Rendall, what makes Friedrich's construction particularly attractive is the use of the Lagrangian description of the fluid flow, as it is known that Lagrangian coordinates are uniquely suited for treating free boundary problems involving the non-relativistic Euler equations, and have in fact been employed to a great success to study them [15, 18, 20, 57].

It remains to be seen whether Friedrich's ideas will lead to satisfactory existence theorems for the Cauchy problem for fluid bodies. But for such a project to be successful, one needs to be able to carry out step (i) as explained above, namely, to use the reduced system of [23] to separately solve Einstein equations in the cases of vacuum and coupled to fluid matter. Only then the passage across the boundary from the Einstein–Euler-Entropy to the vacuum equations can be analyzed. Notice that in the case of vacuum, the notion of a Lagrangian description is somewhat artificial, but it can be given meaning by the choice of a time-like vector field which should agree with the fluid four-velocity once matter is introduced.

Well-posedness of the vacuum Einstein equations using the formalism of [23] has been established in [24, 25] (see also [26–28]). This leaves us with the question of well-posedness of the Einstein–Euler-Entropy system. In other words, we seek a

[b]It should be noticed that even self-gravitating Newtonian fluid bodies are not well understood. In fact, with the exception of the viscous case studied by Secchi [53–55], not many results besides that of Makino [42] seem to be available.

solution to the following:

Establish a well-posedness result for the Einstein–Euler-Entropy system with a general equation of state, by writing the set of equations as a first order symmetric hyperbolic system via a Lagrangian description of the fluid flow, as in [23], *and under the assumption that the fluid fills the entire initial Cauchy surface* Σ, *i.e.* $\varrho_0 \geq c > 0$, *with possibly additional hypotheses consistent with physical requirements.*

This is the problem addressed and solved in this work. Some terminology is needed before we can make a precise statement — see Theorem 2.1 in Sec. 2.2. We stress that Theorem 2.1 had been proven much earlier by Choquet-Bruhat [10] and Lichnerowicz [36]. What is new is the method of proof, employing the Lagrangian description of the fluid flow. It should also be noticed that, while the equations we use are essentially those of [23, 30], to the best of our knowledge, a complete proof of well-posedness relying on Friedrich's techniques is not available in the literature. In fact, while for the trained eye Theorem 2.1 will sound as an expected consequence of the formalism developed by Friedrich, the intricacies of General Relativity teach us to never take something for granted and always welcome full-detailed proofs. We also believe that a wider audience can benefit from our more or less self-contained exposition, which starts from basic definitions and proceeds step-by-step towards the proof of well-posedness.

1.1. *Outline of the paper and notation*

The paper is organized as follows. In Sec. 2, we recall several basic definitions, fix our notation, introduce the main equations and hypotheses, and state the main theorem. In Sec. 3, we introduce the frame formalism and gauge conditions that constitute the basis of Friedrich's method. The gauged or reduced Einstein–Euler-Entropy system is also presented in this section. The proof of the main theorem is carried out in Sec. 4. It consists of three parts: determination of the initial data (Sec. 4.1), well-posedness of the reduced system (Sec. 4.2), and the "propagation of the gauge" (Sec. 4.3). This last step is what guarantees that a solution of the Einstein–Euler-Entropy system in a particular gauge — the reduced system — yields a solution to the original set of equations. Finally, in Sec. 5, we make some closing remarks.

We have chosen to present our arguments in a logical rather than constructive order. This means that instead of starting by showing how the reduced equations are obtained from the Einstein–Euler-Entropy system by means of a gauge choice, we first state the reduced equations, then derive its solutions, and finally show that they correspond to solutions of the original equations of motion. This particular order is adopted because the equations we use in both the reduced and the propagation of the gauge systems are equivalent to those of [23, 30], although we shall write them in a slightly different fashion. Hence, as they have been considered before, there is no immediate need to show how the reduced equations are extracted from the original ones by a suitable choice of gauge. This procedure is briefly presented nonetheless in the appendices for the reader's convenience. Those not familiar with the work [23] are encouraged to read Appendix B prior to Sec. 4.

Notation 1. The Sobolev space between manifolds M and N will be denoted by $H^s(M,N)$. When no confusion can arise, the reference to N will be suppressed, and $H^s(M,N)$ will be written $H^s(M)$, or even H^s when M is clear from the context. H^s_{ul} denotes the uniformly local Sobolev spaces, whose definition is recalled in Appendix A.

2. The Basic Setting and the Main Result

Our main object of study is a four-dimensional Lorentzian manifold $(\mathcal{M}, g)$, called a *space-time*. In General Relativity, we are interested in the sub-class of Lorentzian manifolds where Einstein equations are satisfied. We recall that these are

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \mathscr{K} T_{\alpha\beta}, \tag{2.1}$$

where $R_{\alpha\beta}$ and R are, respectively, the Ricci and scalar curvature of the metric g, $T_{\alpha\beta}$ is the stress-energy tensor which encodes information about the matter fields, and $\mathscr{K} = 8\pi\frac{G}{c^4}$, with c being the speed of light and G Newton's constant.[c] A space-time is called an *Einsteinian space-time* when (2.1) is satisfied. $\mathcal{M}$ will always be assumed to be oriented and time-oriented.

The left-hand side of (2.1) is divergence free as a consequence of the Bianchi identities. Hence, regardless of the particular matter model which is considered, the stress energy tensor has to satisfy

$$\nabla^\alpha T_{\alpha\beta} = 0, \tag{2.2}$$

which then gives equations of motion for the matter fields (see Sec. 2.2 below); ∇ in (2.2) denotes the Levi-Civita connection associated with g. Equation (2.2) is sometimes referred to as the local law of momentum and energy conservation.

Convention 1. Throughout the paper we shall adopt the convention $(+\,-\,-\,-)$ for the metric.

Definition 2.1. A *fluid source* is a triple $(\mathcal{U}, g, u)$, where $(\mathcal{U}, g)$ is a domain of a space-time and u a time-like vector field on $\mathcal{U}$ of unit norm. Physically, the trajectories of u represent the flow lines of matter. Given a fluid source, the *stress-energy*

[c]As it is customary to drop the factor 8π, and we adopt units where $G = c = 1$, we could set $\mathscr{K} = 1$. But since we will be working with a lesser familiar formalism, it is useful to keep the constant $\mathscr{K}$, in which case it is always possible to set $\mathscr{K} = 0$ in order to compare the resulting equations with the well-studied vacuum case.

tensor of a perfect fluid is given by

$$T_{\alpha\beta} = (p + \varrho)u_\alpha u_\beta - p g_{\alpha\beta}, \tag{2.3}$$

where p, called the *pressure of the fluid*, and ϱ, called the *density of the fluid*, are non-negative real valued functions. A *perfect fluid source* is a fluid source together with a stress-energy tensor given by (2.3).

Perfect fluid sources, or perfect fluids for short, are often used to study space-times where a continuous distribution of matter exists (see, e.g., [61]). The assumption that matter is described by a stress-energy tensor of the form (2.3) means that no dissipation of any sort is present; in particular one neglects possible effects due to heat conduction, viscosity or shear stresses.[d]

For perfect fluids, Eq. (2.2) becomes

$$\begin{aligned}\nabla^\alpha T_{\alpha\beta} &= ((p + \varrho)\nabla^\alpha u_\alpha + u^\alpha \nabla_\alpha \varrho)u_\beta \\ &\quad + (p + \varrho)u^\alpha \nabla_\alpha u_\beta + u_\beta u^\alpha \nabla_\alpha p - \nabla_\beta p = 0.\end{aligned} \tag{2.4}$$

Taking the inner product of (2.4) with u and using $u_\alpha u^\alpha = 1$ (which also implies $u^\alpha \nabla_\beta u_\alpha = 0$), gives

$$(p + \varrho)\nabla^\alpha u_\alpha + u^\alpha \nabla_\alpha \varrho = 0, \tag{2.5}$$

known as the *conservation of energy* or *continuity equation*. Then, using (2.5) into (2.4) produces

$$(p + \varrho)u^\alpha \nabla_\alpha u_\beta + u_\beta u^\alpha \nabla_\alpha p - \nabla_\beta p = 0, \tag{2.6}$$

known as *conservation of momentum equation*. Together, Eqs. (2.5) and (2.6) are known as the *Euler equations* for a relativistic fluid.

In physically relevant models, the functions p and ϱ are not independent but are related by what is called an *equation of state*, which is an (usually smooth) invertible function

$$p = p(\varrho). \tag{2.7}$$

Fluids where (2.7) is satisfied are called *barotropic* fluids, with the particular case $p \equiv 0$ called *dust* or *pressure-free matter*. Physical situations of interest where barotropic fluids are employed include some models of cold (more precisely, zero temperature) matter, such as completely degenerate cold neutron gases (which are used to model nuclear matter in the interior of neutron stars) [2, 8, 56]; the so-called ultra-relativistic fluids, i.e. fluids in thermal equilibrium where the energy (which in relativistic terms is described by ϱ due to the equivalence of mass and energy) is largely dominated by radiation [2, 61]; and fluids of electron-positron pairs [61]. Barotropic fluids are also important in Cosmology, where several models of the early

[d]In fact, the (here postulated) stress-energy tensor (2.3), can be motivated from basic physical principles, which can be taken as characterizing ideal fluids; see, e.g., [2, 21].

universe assume that the distribution of matter is described by an ultra-relativistic fluid [2, 8, 61, 62].

As usual in General Relativity, space-time itself is a dynamic quantity, therefore its geometry, encoded in the metric g, as well as the the dynamics of other fields present on space-time, have to be determined as solutions to the Einstein equations (2.1) coupled to matter via (2.2). In particular, the fluid source $(\mathcal{U}, g, u)$ is not a given, but has to arise from the solutions of the coupled system. The *Einstein–Euler system for barotropic fluids*, or *barotropic Einstein–Euler system*, is given by Eqs. (2.1), (2.5), (2.6), and (2.7), with $T_{\alpha\beta}$ given by (2.3), and subject to the constraint $u^\alpha u_\alpha = 1$. The unknowns to be determined are the metric g, the four-velocity u, and the matter density ϱ.

The Einstein–Euler system for barotropic fluids has been studied by many authors. In fact, most of the results cited in the introduction deal with this case. The interested reader can consult the papers [3, 16, 31, 33, 43, 48, 49] and the monographes [2, 8, 36], as well as the references therein.

Similar to the well-known case of vacuum Einstein equations [9], when addressing the solvability of the barotropic Einstein–Euler system, one has to investigate its constraints and make suitable choices for the spaces where solutions will be sought. Since we shall deal with the more general case of fluids that are not necessarily barotropic, we postpone this discussion for the time being, turning our attention first to some thermodynamic tools that will be necessary in the sequel. We shall return to the barotropic case in Sec. 5.

2.1. *Thermodynamic properties of perfect fluids*

There are important scenarios where it is known that the pressure is not determined by the matter density only. These include the so-called polytropic fluids, which are used in several stellar models (see, e.g., [2, 8, 61, 64] and references therein). In such cases, (2.7) has to be replaced by a more general equation of state involving other physical quantities, which are usually assumed to be of thermodynamic nature, as we now describe.

As in many situations in General Relativity, it is important to identify those quantities measured by a local inertial observer. Let r be the *rest mass (or energy)*[e] *density*,[f] defined as the mass density measured in the local rest frame. It is assumed that this quantity obeys the conservation law

$$\nabla_\alpha (r u^\alpha) = 0, \tag{2.8}$$

which states that mass is (locally) conserved. Notice that on dimensional grounds, $\frac{1}{r}$ is the specific (i.e. per unit of mass) volume. The difference between the mass

[e]Due to the equivalence of mass and energy, and our choice of units with $c = 1$, we shall use the terms mass and energy interchangeably.

[f]Also called *particle number density* [30], *baryon number density* [2] or yet *proper material density* [36].

density ϱ and the rest mass density r is by definition the *specific internal energy* ϵ (as measured in the local rest frame):

$$\varrho = r(1+\epsilon). \tag{2.9}$$

Therefore, as p, r and ϱ, ϵ is a real valued function on $\mathcal{U} \subseteq \mathcal{M}$. In the context of relativistic fluids, the relation (2.9) has been first introduced by Taub [59] and has been widely used since [2, 8, 30, 36].

For non-barotropic fluids, further relations among the variables of the problem have to be introduced in order to have a well-determined system of equations. It is natural to assume that the first law of thermodynamics holds, i.e.

$$d\epsilon = K\,ds - p\,dv,$$

where K is the absolute temperature, s the specific entropy, and v the specific volume. These are all non-negative real valued functions on $\mathcal{U}$. In light of (2.9) and $v = \frac{1}{r}$, the first law can be written as[g]

$$d\varrho = \frac{p+\varrho}{r}\,dr + rK\,ds. \tag{2.10}$$

We now turn to the generalization of (2.7), i.e. to the appropriate equation of state that has to be provided. Although we presently have seven thermodynamic variables, namely, p, ϱ, r, ϵ, K, v and s, the above equations imply relations among them. In fact, for a perfect fluid, only two of such quantities are independent [2], with the remaining ones determined by relations depending exclusively on the nature of the fluid. On physical grounds, we should assume that such relations are invertible, what renders the question of which two thermodynamical quantities are the independent ones a matter of preference. We shall assume henceforth that r and s are independent. We thus postulate an equation of state of the form

$$\varrho = \mathscr{P}(r,s), \tag{2.11}$$

where $\mathscr{P}$ is a given smooth function, invertible in the sense that we can solve for $r = r(\varrho, s)$ and $s = \varrho(\varrho, r)$. From (2.10) and (2.11), we then obtain

$$p = r\frac{\partial \varrho}{\partial r} - \varrho, \tag{2.12}$$

and

$$K = \frac{1}{r}\frac{\partial \varrho}{\partial s}. \tag{2.13}$$

The validity of (2.10) along with (2.5) implies

$$K u^\alpha \nabla_\alpha s = -\frac{1}{r^2}(p+\varrho)\nabla_\alpha(r u^\alpha). \tag{2.14}$$

[g]The case of interest in this paper is when the matter density does not vanish, hence $\frac{1}{r}$ is well defined because of (2.9). The case of vanishing ϱ is, however, important, as explained in the introduction.

Therefore, in light of (2.8), and assuming that the temperature is not zero (which is consistent with (2.13) and (2.11)),

$$u^\alpha \nabla_\alpha s = 0. \tag{2.15}$$

In other words, the entropy is conserved along the flow lines of the fluid. When (2.15) holds, the motion of the fluid is said to be *locally adiabatic*. A fluid is said to be *isentropic* if $s =$ constant, and *non-sentropic* otherwise.

Remark 2.1. If (2.8) is not assumed, then from (2.14) it only follows that

$$\nabla_\alpha (r u^\alpha) \leq 0,$$

and

$$u^\alpha \nabla_\alpha s \geq 0.$$

Although these two inequalities have a clear physical interpretation — from the point of view of inertial observers the rest mass cannot increase and the entropy cannot decrease — if (2.8), and therefore (2.15), is not assumed, the motion of the fluid is underdetermined.

The first law of thermodynamics, Eq. (2.10), will always be assumed to hold, therefore, because of (2.14), we shall work with (2.15) rather than (2.8) to form our system of equations.

2.2. *The Einstein–Euler-Entropy system*

The *Einstein–Euler-Entropy system* is the system comprised of Eqs. (2.1), (2.5), (2.6), (2.11), (2.12), (2.15), subject to the constraint $u^\alpha u_\alpha = 1$, with $T_{\alpha\beta}$ given by (2.3), and $\mathscr{P}$ in (2.11) a given smooth invertible function. The unknowns to be determined are the metric g, the four-velocity u, the rest mass density r, and the specific entropy s.

For previous works on the Einstein–Euler-Entropy system, the reader can consult the monographes [2, 8, 36].

Our focus here is on the Cauchy problem, therefore we need to state what the initial data are. This should consist of the usual initial data for the Einstein equations and initial data for the matter fields. These have to satisfy suitable constraint equations, as we now recall. Because of our signature convention, the metric g_0 on the initial three slice Σ is negative-definite, a fact which we stress by calling the pair (Σ, g_0) a *negative Riemannian manifold*.

Definition 2.2. A *pre-initial data set for the Einstein equations*, or *pre-initial data set* for short, is a triple (Σ, g_0, κ), where (Σ, g_0) is a three-dimensional negative Riemannian manifold and κ a symmetric two-tensor on Σ. A *development* of a pre-initial data set (Σ, g_0, κ) is a space-time $(\mathcal{M}, g)$ which admits an isometric embedding of (Σ, g_0), with κ being the second fundamental form of the embedding. A development $(\mathcal{M}, g)$ is called an *Einsteinian development* if $(\mathcal{M}, g)$ is an Einsteinian space-time, i.e. Einstein equations are satisfied on $\mathcal{M}$.

Let $(\mathcal{M}, g)$ be an Einsteinian development of (Σ, g_0, κ). Then the fact that Σ is embedded into $\mathcal{M}$ with second fundamental form κ and Einstein equations are satisfied implies that the following identities hold[h] on Σ,

$$R_{g_0} - |\kappa|^2_{g_0} + (\operatorname{tr}_{g_0} \kappa)^2 + 2\mu = 0, \tag{2.16}$$

known as the *Hamiltonian constraint*, and

$$\operatorname{div}_{g_0}(\kappa - (\operatorname{tr}_{g_0} \kappa) g_0) + J = 0, \tag{2.17}$$

known as the *momentum constraint*, with μ and J defined by

$$\mu := \mathscr{K} T(n, n), \tag{2.18}$$

and

$$J := \mathscr{K} T(n, \cdot), \tag{2.19}$$

where n is the unit normal of Σ inside $\mathcal{M}$, T is the stress-energy tensor, and $T(n, \cdot)$ is viewed as a one-form on Σ. In the above, R_{g_0}, $|\cdot|_{g_0}$, tr_{g_0}, div_{g_0} are, respectively, the scalar curvature, the pointwise norm, the trace, and the divergence, all with respect to the metric g_0. Notice that Definition 2.2 and Eqs. (2.16)–(2.19) are general, in the sense that they do not assume that T is the stress-energy tensor of a perfect fluid. When T has the form (2.3), then (2.18) and (2.19) become

$$\mu = \mathscr{K}(p + \varrho)(1 - |\pi(u)|^2_{g_0}) - \mathscr{K} p, \tag{2.20}$$

and[i]

$$J = \mathscr{K}(p + \varrho)\sqrt{1 - |\pi_g(u)|^2_{g_0}}\, \pi_g(u), \tag{2.21}$$

where $\pi_g : T\mathcal{M}|_\Sigma \to T\Sigma$ is the orthogonal projection onto the tangent bundle of Σ.

From (2.20) and (2.21), it is seen that, additionally to (Σ, g_0, κ), to solve the Einstein–Euler-Entropy system, one has to prescribe ϱ, p, and a vector field v (which will satisfy $v = \pi(u)$ once solutions are obtained). However, as we adopt the point of view that the independent thermodynamic variables are s and r, we do not prescribe p and ϱ directly. Rather, r and s are given as initial data, and then ϱ and p are determined on Σ by (2.11) and (2.12), respectively.

Definition 2.3. An *initial data set for the Einstein–Euler-Entropy system* is a 7-uple $(\Sigma, g_0, \kappa, r_0, \varsigma_0, v, \mathscr{P})$, where (Σ, g_0) is a three-dimensional negative Riemannian manifold endowed with a symmetric two-tensor κ; r_0 and ς_0 are non-negative

[h]See, e.g., [32] for a proof. The plus sign on μ is due to our signature convention; some authors define J with a negative sign.

[i]As usual we identify vectors and co-vectors, since J is given as a vector in (2.21) but as a one form in (2.19).

real valued functions[j] $r_0, \varsigma_0 : \Sigma \to \mathbb{R}_+$; $\mathscr{P} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ is a smooth invertible function as in (2.11); and v is a vector field on Σ; such that the *constraint equations* (2.16) and (2.17) are satisfied, with μ and J given by

$$\mu = \mathscr{K}(p_0 + \varrho_0)(1 - |v|^2_{g_0}) - \mathscr{K} p_0,$$

and

$$J = \mathscr{K}(p_0 + \varrho_0)\sqrt{1 - |v|^2_{g_0}}\, v,$$

where $\varrho_0 = \varrho(r_0, \varsigma_0)$ and $p_0 = p_0(r_0, \varsigma_0)$ are given by (2.11) and (2.12), respectively.

In order to state the main result, we need to impose further conditions that are either of physical nature or are required in order to apply certain analytic techniques.

One quantity of physical relevance in non-relativistic fluid dynamics is the speed of acoustic waves. For relativistic fluids, we define the *sound speed* ν by

$$\nu^2 = \left(\frac{\partial p}{\partial \varrho}\right)_s = \frac{r}{p + \varrho}\frac{\partial p}{\partial r}, \tag{2.22}$$

where, as usual in Thermodynamics, $(\frac{\partial}{\partial x})_y$ means treating y as a constant when taking derivatives with respect to x. ν^2 is well defined in that, for physically relevant equations of state, pressure cannot decrease as ϱ increases. The fact that (2.22) is correctly interpreted as the speed of acoustic waves follows from analyzing the characteristic manifolds of the system formed by Eqs. (2.5), (2.6), (2.15) and (2.11) with a given background metric; we refer the reader to [2, 36] for details.

We shall require that the following inequalities hold:

$$\nu^2 > 0, \tag{2.23a}$$

$$\nu^2 \leq 1, \tag{2.23b}$$

$$\frac{p + \varrho}{r} > 0. \tag{2.23c}$$

(2.23a) and (2.23b) assert the physical requirements that the sound speed is positive and does not exceed the speed of light.[k] The interpretation of (2.23c) is that *specific enthalpy* of the fluid, defined as $\frac{p+\varrho}{r}$, is positive. In view of (2.22), it should be noticed that (2.23a) together with (2.23c) excludes the possibility of pressure-free matter. Our techniques can be easily adapted to allow such a case nonetheless, leading to a theorem similar to Theorem 2.1 below (see Sec. 5).

[j] We write ς_0 instead of s_0 for the entropy at time zero to avoid confusion with the quantity s_α introduced below.

[k] It is not difficult to see that equality in (2.23b) happens if, and only if, the four-divergence of u vanishes, i.e. $\nabla_\alpha u^\alpha = 0$ (see, e.g., [8]). When this happens, the fluid is called *incompressible*. Recall that incompressibility is defined in non-relativistic physics as the vanishing of the three-velocity, in which case sound waves travel with infinity speed.

When the initial slice Σ is compact and the functions involved continuous, inequalities (2.23a) and (2.23c) automatically imply that on Σ

$$\nu^2 \geq c \quad \text{and} \quad \frac{p+\varrho}{r} \geq c, \tag{2.24}$$

for some constant $c > 0$. These conditions are needed to apply certain existence results for first-order symmetric hyperbolic systems. When Σ is non-compact, our techniques still apply, but without a bound of the form (2.24) and further assumptions on the initial data, the size of the time interval where the solution exists can tend to zero near the asymptotic region of Σ. We therefore assume (2.24), briefly commenting on more general situations in Sec. 5. Also, for Σ non-compact, in order to accommodate conditions at infinity which are not too restrictive, we shall employ the uniformly local Sobolev spaces[1] H^s_{ul} originally introduced by Kato [34]. Their definition and basic properties are recalled in Appendix A; although it comes as no surprise that H^s_{ul} and H^s are equivalent (as Banach spaces) when Σ is compact. It is assumed that the metric employed to define the spaces $H^s_{ul}(\Sigma)$ is equivalent, up to the sign convention, to the metric g_0.

Yet another difficulty which arises in the non-compact case is that u can become arbitrarily close to the boundary of the light-cone in the asymptotic region. This would lead to a breakdown of the positivity of certain matrices required for our proofs, again implying that the time interval on which the solutions exist cannot be made uniform. To prevent this, we impose a bound on the size of the initial tangent velocity v. To see that such a bound gives the desired control over u near the initial hypersurface, simply notice that in normal coordinates at a point p on the hypersurface with $\frac{\partial}{\partial x^0}$ normal to Σ, the set of unit length future-directed vectors on $T_p\mathcal{M}$ is given as usual by the upper sheet of the time-like hyperboloid which is asymptotic to the light-cone.

We are now in a position to state our main result.

Theorem 2.1. *Let $(\Sigma, g_0, \kappa, r_0, \varsigma_0, v, \mathscr{P})$ be an initial data set for the Einstein–Euler-Entropy system, with $g_0 \in H^{s+1}_{ul}(\Sigma), \kappa, v, r_0, \varsigma_0 \in H^s_{ul}(\Sigma)$, where $s > \frac{3}{2} + 2$. Suppose further that $\varsigma_0 \geq 0, r_0 \geq c_1, \frac{p_0+\varrho_0}{r_0} \geq c_1, \nu_0^2 = \nu^2(r_0, \varsigma_0) \geq c_1$, and $|v|_{g_0} \geq -c_2$, for some constants c_1, $c_2 > 0$. Then there exists an Einsteinian development $(\mathcal{M}, g)$ of $(\Sigma, g_0, \kappa, r_0, \varsigma_0, v, \mathscr{P})$ which is a perfect fluid source, with $\mathcal{M}$ diffeomorphic to $[0, T_E] \times \Sigma$ for some real number $T_E > 0$. Moreover, $g \in C^0([0, T_E], H^{s+1}_{ul}(\Sigma)) \cap C^1([0, T_E], H^s_{ul}(\Sigma)) \cap C^2([0, T_E], H^{s-1}_{ul}(\Sigma))$, the matter density ϱ and the pressure p of the fluid source are given by* (2.11) *and* (2.12), *respectively, where the functions r and s in* (2.11)–(2.12) *belong to $C^0([0, T_E], H^s_{ul}(\Sigma)) \cap C^1([0, T_E], H^{s-1}_{ul}(\Sigma))$, satisfy $r > 0, s \geq 0$, and are such that $r|_\Sigma = r_0, s|_\Sigma = \varsigma_0$, and $\nu^2 = \nu^2(r, s) > 0$. Furthermore, denoting by u the unit time-like vector field of the fluid source, we*

[1]Other function spaces can be used. In particular, the "little ℓ_p-Sobolev spaces", $\ell_p(H^s)$, could be employed, these being, in fact, better suited for treating more general data on non-compact manifolds, see, e.g., [60].

have that $u \in C^0([0,T_E], H^s_{ul}(\Sigma)) \cap C^1([0,T_E], H^{s-1}_{ul}(\Sigma))$ *and* $\pi_g(u) = v$, *where* $\pi_g : T\mathcal{M}|_\Sigma \to T\Sigma$ *is the orthogonal projection onto the tangent bundle of* Σ.

Remark 2.2. Notice that the pointwise inequalities of Theorem 2.1 all make sense in that $s > \frac{3}{2} + 2$ implies that the quantities involved are continuous. As discussed above, when Σ is compact, the hypotheses involving c_1 can be relaxed by assuming that those quantities are simply greater than zero, H^s_{ul} can be replaced by the ordinary Sobolev spaces and the bound on $|v|_{g_0}$ is automatically satisfied (recall that g_0 is negative definite).

3. The Einstein–Euler-Entropy System in the Frame Formalism

In this section, we shall use the so-called frame formalism to write a different set of equations for the Einstein–Euler-Entropy system. It will be shown in Sec. 4 that solutions to the new system imply existence of solutions to the original Einstein–Euler-Entropy equations. As in most analytic investigations of the Einstein equations, the point of view is essentially local as a consequence of the phenomenon of finite propagation speed. Thus a chart U should be implicitly understood whenever coordinates are involved. If Σ is a space-like three surface, or a *slice* for short, then we shall use a slight abuse of notation and still denote by Σ the set $\Sigma \cap U$.

In the frame formalism, the information about the metric is encoded in an orthonormal frame $\{e_\mu\}_{\mu=0}^3$, where the frame is related to the coordinate basis by

$$e_\mu = e_\mu^A \frac{\partial}{\partial x^A}. \tag{3.1}$$

Notation 2. From now on, unless otherwise specified, all tensor fields will be expressed in the orthonormal frame $\{e_\mu\}_{\mu=0}^3$, with Greek letters used to indicate the components of such fields in the basis $\{e_\mu\}_{\mu=0}^3$. A bar over an index, e.g., $\bar{\alpha}$, indicates that it can take only the values $1, 2$ or 3, with summation of barred indices being only over $1, 2, 3$ as well. The only exception for our choice of basis will be for the frame itself, which will be given in terms of the frame coefficients e_μ^A in (3.1). As in (3.1), capital Latin letters range from 0 to 3 and will be used to denote components with respect to the coordinate basis; a bar, e.g., $\bar{A}$, indicates restriction to $1, 2$ or 3. Indices are still raised and lowered with the metric g, as usual.

By construction, in terms of the orthonormal frame $\{e_\mu\}_{\mu=0}^3$, the metric g is always represented by the Minkovski metric $g_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$. The relation to the metric in the basis $\{\frac{\partial}{\partial x^A}\}_{A=0}^3$ is given by

$$g^{AB} = e_\alpha^A e_\beta^B g^{\alpha\beta}. \tag{3.2}$$

The *connection coefficients* $\Gamma_{\alpha\beta}^\gamma$ are defined via

$$\nabla_\alpha e_\beta = \Gamma_{\alpha\beta}^\gamma e_\gamma. \tag{3.3}$$

The condition that ∇ is compatible with the metric then takes the form

$$\Gamma^{\mu}_{\alpha\beta} g_{\mu\gamma} + \Gamma^{\mu}_{\alpha\gamma} g_{\mu\beta} = 0. \tag{3.4}$$

One of the key ingredients of the formalism we shall employ is to treat the connection coefficients as unknowns in their own right. As $\Gamma^{\gamma}_{\alpha\beta}$ is given in terms of first derivatives of the metric, treating them as independent variables allows us to express the original Einstein–Euler-Entropy system, which involves second derivatives of g, as a first order system.[m]

Symmetry (3.4) will be assumed throughout. In other words, we shall only need evolution equations for 24 independent Γ's[n] with the remaining components explicitly defined via (3.4).

Since it is not possible to decide from the connection coefficients alone whether ∇ is the Levi-Civita connection of g [51], further conditions will be necessary. The requirement that the connection is torsion-free, along with the (once contracted) Bianchi identities and the standard decomposition of the Riemann curvature tensor in terms of the Weyl and Schouten tensors, will be imposed as further equations of motion of the system. In this regard, we introduce the tensor $d^{\alpha}_{\beta\gamma\delta}$ (d for "decomposition") defined as

$$d^{\alpha}_{\beta\gamma\delta} := R^{\alpha}_{\beta\gamma\delta} - W^{\alpha}_{\beta\gamma\delta} - g^{\alpha}_{[\gamma} S_{\delta]\beta} + g_{\beta[\gamma} S^{\alpha}_{\delta]}, \tag{3.5}$$

where $[\alpha\,\beta]$ means that the indices are anti-symmetrized; $W^{\alpha}_{\beta\gamma\delta}$ is the Weyl tensor; $S_{\alpha\beta}$ is the Schouten tensor, given by

$$S_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{6} R g_{\alpha\beta}, \tag{3.6}$$

and $R^{\alpha}_{\beta\gamma\delta}$ is the Riemann tensor, which can be written as

$$\begin{aligned} R^{\alpha}_{\beta\gamma\delta} = {} & e_\gamma(\Gamma^{\alpha}_{\delta\beta}) - e_\delta(\Gamma^{\alpha}_{\gamma\beta}) - \Gamma^{\alpha}_{\mu\beta}(\Gamma^{\mu}_{\gamma\delta} - \Gamma^{\mu}_{\delta\gamma}) + \Gamma^{\alpha}_{\gamma\mu}\Gamma^{\mu}_{\delta\beta} \\ & - \Gamma^{\alpha}_{\delta\mu}\Gamma^{\mu}_{\gamma\beta}. \end{aligned} \tag{3.7}$$

By construction, $d^{\alpha}_{\beta\gamma\delta}$ possesses the usual symmetries of the Riemann tensor, since such symmetries are shared by $W^{\alpha}_{\beta\gamma\delta}$ and $-g^{\alpha}_{[\gamma} S_{\delta]\beta} + g_{\beta[\gamma} S^{\alpha}_{\delta]}$. Recall that the *torsion* of the connection is the tensor $\mathcal{T}$ defined via

$$\mathcal{T}^{\mu}_{\alpha\beta} e_\mu = -[e_\alpha, e_\beta] + (\Gamma^{\mu}_{\alpha\beta} - \Gamma^{\mu}_{\beta\alpha}) e_\mu,$$

[m]This argument only presents the basic heuristic intuition of the method. In fact, although the metric will not be one of the basic unknowns, our system would be third-order in the metric if written in terms of it. This is because the system will involve first derivatives of the Weyl tensor, see Eqs. (3.9), (3.10) and (3.14e). It should be noticed that this is not an oddity of our formalism. If one tries to approach the Einstein–Euler-Entropy system in the usual formalism of second order equations, it cannot be directly solved as it stands; a quasi-diagonalization process has then to be carried out, leading to a system that is also third-order in the metric [10].

[n]Due to (3.4), there are $\frac{n^2(n-1)}{2}$ independent components of $\Gamma^{\gamma}_{\alpha\beta}$ in n space-time dimensions, $n = 4$ in our case.

where $[\cdot,\cdot]$ is the usual commutator of two vector fields. Notice that

$$\mathcal{T}^{\mu}_{\alpha\beta} = -\mathcal{T}^{\mu}_{\beta\alpha}. \tag{3.8}$$

Aiming at the Bianchi identities, we define the *Friedrich tensor* by

$$F^{\alpha}_{\beta\gamma\delta} := W^{\alpha}_{\beta\gamma\delta} - g^{\alpha}_{[\gamma} S_{\delta]\beta}, \tag{3.9}$$

and let

$$F_{\alpha\beta\gamma} := \nabla_{\mu} F^{\mu}_{\alpha\beta\gamma}. \tag{3.10}$$

To understand the role of $F_{\alpha\beta\gamma}$, consider for simplicity the case of vacuum. Then the once-contracted Bianchi identity

$$\nabla_{\mu} R^{\mu}_{\alpha\beta\gamma} = \nabla_{\beta} R_{\alpha\gamma} - \nabla_{\gamma} R_{\alpha\beta},$$

combined with Einstein equations (2.1) yields $F_{\alpha\beta\gamma} = 0$ whenever the decomposition of the Riemann tensor holds, i.e. $d^{\alpha}_{\beta\gamma\delta} = 0$.

The last definition we need to introduce is

$$q_{\alpha} := (p+\varrho) u^{\mu} \nabla_{\mu} u_{\alpha} - \nu^2 (p+\varrho) u_{\alpha} \nabla_{\mu} u^{\mu} - \nabla_{\alpha} p. \tag{3.11}$$

q_{α} will replace (2.6) in the new set of equations. To motivate this, suppose that we have a solution to the Einstein–Euler-Entropy system. Then differentiating (2.11) in the direction of u, using (2.12), (2.8), (2.15), and (2.22) yield

$$u^{\alpha} \nabla_{\alpha} p = -(p+\varrho)\nu^2 \nabla_{\alpha} u^{\alpha}.$$

Using this into (2.6) then gives $q_{\alpha} = 0$.

Tracing Einstein equations (2.1) gives

$$R = -\mathscr{K} T, \tag{3.12}$$

where T is the trace of the stress-energy tensor. Using (2.1) and (3.12) into (3.6) produces

$$S_{\alpha\beta} = \mathscr{K}\left(T_{\alpha\beta} - \frac{1}{3} T g_{\alpha\beta}\right), \tag{3.13}$$

which is an equivalent way of writing the Einstein equations.

We can now define the *Einstein–Euler-Entropy system in the frame formalism* as the system comprised of

$$\begin{cases} \text{Eqs. (2.5), (2.8), (2.11), (2.12), (2.15), (2.22), (3.13),} & \text{(3.14a)} \\ q_{\alpha} = 0, & \text{(3.14b)} \\ \mathcal{T}^{\mu}_{\alpha\beta} = 0, & \text{(3.14c)} \\ d^{\alpha}_{\beta\gamma\delta} = 0, & \text{(3.14d)} \\ F_{\alpha\beta\gamma} = 0, & \text{(3.14e)} \end{cases}$$

subject to the constraint $u^{\alpha} u_{\alpha} = 1$, with $T_{\alpha\beta}$ given by (2.3), and where q_{α}, $\mathcal{T}^{\mu}_{\alpha\beta}$, $d^{\alpha}_{\beta\gamma\delta}$ and $F_{\alpha\beta\gamma}$ are given by (3.11), (3.8), (3.5), and (3.10), respectively. The unknowns to be determined are the frame coefficients e^A_α, the connection coefficients $\Gamma^\gamma_{\alpha\beta}$, the Weyl tensor $W^\alpha_{\beta\gamma\delta}$, the four-velocity u, the rest mass density r, the specific entropy s, and the matter density ϱ. The usual symmetries of $W^\alpha_{\beta\gamma\delta}$, and those of $\Gamma^\gamma_{\alpha\beta}$ determined by (3.4), are explicitly assumed, therefore the system is written for only the 20 independent components of the Weyl tensor.[o] and the 24 independent connection coefficients.[p]

Remark 3.1. Obviously, when attempting to solve (3.14), it is not yet known that W is the Weyl tensor. In particular, the trace-free condition must be demonstrated.

One way to motivate the choice of the Weyl tensor as one of the unknowns is to consider the simpler case of vacuum. Then, the validity of $d^\alpha_{\beta\gamma\delta} = 0$ gives $R^\alpha_{\beta\gamma\delta} = W^\alpha_{\beta\gamma\delta}$ which, upon contraction, produces the vacuum Einstein equations. Since the Bianchi identities are necessary conditions for the solvability of Einstein equations, we also have $F_{\beta\gamma\delta} = \nabla_\alpha W^\alpha_{\beta\gamma\delta} = \nabla_\alpha R^\alpha_{\beta\gamma\delta} = 0$ in the case of vacuum. We see in this way that the usual vacuum Einstein equations can be recovered from the Weyl tensor.[q]

Equation (3.14e) is sometimes referred to as the "Bianchi equation", and its importance in General Relativity has been long recognized. It figures in the Newman–Penrose's spin formalism and has been used in the study of massless fields [45], including gravitational radiation [50]. Friedrich has employed it extensively, initially to obtain energy estimates in terms of the Bel–Robinson tensor [26–28], and later to derive several hyperbolic reductions for the Einstein equations, both in the vacuum case [24, 25] and coupled to matter [23, 30], with some of these results extended in [46]. Friedrich and Nagy also used (3.14e) in their study of the initial-*boundary* value problem in General Relativity [29]. Using a different point of view than Friedrich, the Bianchi equation was used to derive first order symmetric hyperbolic reduced Einstein equations that treat the Riemann tensor as one of the unknowns by Anderson, Choquet-Bruhat and York, both in the vacuum and matter cases[r] [1, 11]. Finally, Eq. (3.14e) has also been employed (in a rather elaborated fashion though) by Christodoulou and Klainerman in their impressive proof of the global nonlinear stability of Minkovski space [12].

[o]In n space-time dimensions, $W^\alpha_{\beta\gamma\delta}$ has $\frac{1}{12}n(n+1)(n+2)(n-3)$ independent components. As explained in Remark 3.1, the trace-free condition is not assumed in the system, thus there is an extra freedom of $\frac{1}{2}n(n+1)$ components.

[p]All the symmetries and which components enter in the system are described in Sec. 3.2, where we also write the system in a more explicit form.

[q]Other choices of variables are, of course, possible. Choquet-Bruhat and York derived a different system — also based on the Bianchi identities — where the Riemann tensor is one of the unknowns of the problem [11].

[r]In [11], after deriving a reduced first-order symmetric hyperbolic system of equations, the authors do not attempt to show the propagation of the gauge. This has been demonstrated in [19] without coupling of the entropy, giving in this way yet another proof of short-time existence for the Einstein–Euler system.

3.1. *Gauge fixing*

The system (3.14) is overdetermined. In order to obtain a reduced system, which is determined and hyperbolic in a precise sense, a gauge choice has to be made. A specific decomposition of the Weyl tensor, suitable for our gauge choice, will also be necessary.

Put

$$\pi_{\alpha\beta} := g_{\alpha\beta} - u_\alpha u_\beta,$$

so $\pi_{\alpha\beta}$ is the metric induced on the space orthogonal to u, with projection given by

$$\pi_\alpha^\beta = g_\alpha^\beta - u_\alpha u^\beta$$

(recall that indices are always raised with $g_{\alpha\beta}$).

Remark 3.2. Despite the terminology, in general, $\pi_{\alpha\beta}$ and π_α^β will not agree with the metric and projection on the $t =$ constant space-like slices, because such slices are not expected to be orthogonal to u except in some special cases. In particular, $\pi_{\alpha\beta}|_\Sigma$ does not, in general, agree with g_0.

Letting $\varepsilon_{\alpha\beta\gamma\delta}$ be the totally anti-symmetric tensor, with the usual convention $\varepsilon_{0123} = +1$, we define

$$\varepsilon_{\alpha\beta\gamma} := \varepsilon_{\mu\nu\sigma\tau} u^\mu \pi_\alpha^\nu \pi_\beta^\sigma \pi_\gamma^\tau.$$

Expanding out, it is easy to obtain

$$\varepsilon^{\mu\alpha\beta}\varepsilon_{\mu\gamma\delta} = -2\pi_{[\gamma}^\alpha \pi_{\delta]}^\beta, \tag{3.15}$$

and

$$\varepsilon^{\mu\nu\alpha}\varepsilon_{\mu\nu\beta} = -2\pi_\beta^\alpha. \tag{3.16}$$

Recall that an observer moving with a (time-like) 4-velocity V measures an electric and a magnetic field given by $\mathcal{F}_{\alpha\beta}V^\beta$ and $\mathcal{F}^*_{\alpha\beta}V^\beta$, respectively, where $\mathcal{F}$ is the Maxwell stress tensor and $\mathcal{F}^*$ its dual. By analogy, it is customary to introduce the following.

Definition 3.1. Let $W_{\alpha\beta\gamma\delta}$ be the Weyl tensor and $W^*_{\alpha\beta\gamma\delta}$ its dual, given by

$$W^*_{\alpha\beta\gamma\delta} = \frac{1}{2}\varepsilon^{\mu\nu}_{\gamma\delta} W_{\alpha\beta\mu\nu}.$$

The *u-electric* and *u-magnetic parts* of $W^\alpha_{\beta\gamma\delta}$ are defined by

$$E_{\alpha\beta} := W_{\mu\nu\sigma\tau} u^\mu u^\sigma \pi_\alpha^\nu \pi_\beta^\tau,$$

and

$$B_{\alpha\beta} := W^*_{\mu\nu\sigma\tau} u^\mu u^\sigma \pi_\alpha^\nu \pi_\beta^\tau,$$

respectively.

It follows that $E_{\alpha\beta}$ and $B_{\alpha\beta}$ are symmetric and trace-free. With the help of (3.15) and (3.16), it is not difficult to verify the following.

Lemma 3.1. *The following decompositions of the Weyl tensor and its dual hold:*

$$\begin{aligned} W_{\alpha\beta\gamma\delta} &= 2(\pi_{\beta[\gamma}E_{\delta]\alpha} - u_\beta u_{[\gamma}E_{\delta]\alpha} - \pi_{\alpha[\gamma}E_{\delta]\beta} + u_\alpha u_{[\gamma}E_{\delta]\beta}) \\ &\quad - 2(u_{[\gamma}B_{\delta]\mu}\varepsilon^\mu_{\alpha\beta} + u_{[\alpha}B_{\beta]\mu}\varepsilon^\mu_{\gamma\delta}), \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} W^*_{\alpha\beta\gamma\delta} &= 2u_{[\alpha}E_{\beta]\mu}\varepsilon^\mu_{\gamma\delta} - 4E_{\mu[\alpha}\varepsilon^\mu_{\beta][\gamma}u_{\delta]} - 4u_{[\alpha}B_{\beta][\gamma}u_{\delta]} \\ &\quad - B_{\mu\nu}\varepsilon^\mu_{\alpha\beta}\varepsilon^\nu_{\gamma\delta}. \end{aligned} \tag{3.18}$$

Decompositions (3.17) and (3.18) allow the Weyl tensor to be eliminated from the system (3.14) in favor of its electric and magnetic components, producing yet another set of equations where $E_{\alpha\beta}$ and $B_{\alpha\beta}$ will be unknowns; see Sec. 3.2.

Until now, $\{e_\mu\}_{\mu=0}^3$ has been an arbitrary frame with respect to which all tensor fields have been written. In particular, there is no relation so far between u and the frame other than the general fact that u can be decomposed in this base, i.e. $u = u^\mu e_\mu$. A *gauge choice* in our formalism will be a specific choice of frame — very much in the same way that a choice of gauge in the coordinate formalism corresponds to a determined choice of coordinates, e.g., wave coordinates.

Definition 3.2. Let $(\mathcal{U}, g, u)$ be a fluid source. An orthonormal frame $\{e_\mu\}_{\mu=0}^3$ in $\mathcal{U}$ is called a *fluid source gauge* if it satisfies

$$e_0 = u, \tag{3.19}$$

with the remaining $\{e_\mu\}_{\mu=1}^3$ being *Fermi propagated* along e_0. By definition, this means

$$\langle \nabla_{e_0} e_{\bar\alpha}, e_{\bar\beta}\rangle = 0, \tag{3.20}$$

for $\bar\alpha, \bar\beta = 1, 2, 3$ (recall our conventions in Notation 2).

Condition (3.20) means that $\{e_{\bar\alpha}\}$ remains orthogonal along the time-flow of u. Roughly speaking, this can be understood as similar to parallel transport, with the important difference that $\{e_{\bar\alpha}\}$ does not remain parallel but is allowed to "rotate about time-axis given by u".

Let t be a parameter for the flow lines of u and Σ be a slice. Then there exists a foliation of a neighborhood of Σ by leaves $\Sigma_t = \{t = \text{constant}\}$, which are space-like and diffeomorphic to Σ. Let $\{\frac{\partial}{\partial x^{\bar A}}\}_{\bar A=1}^3$ be tangent vectors on Σ associated with coordinates $\{x^{\bar A}\}_{\bar A=1}^3$. The coordinates $\{x^{\bar A}\}_{\bar A=1}^3$ can be dragged along u to give coordinates on Σ_t, and since u is time-like, $u \notin \text{span}\{\frac{\partial}{\partial x^{\bar A}}\}$. Setting $\frac{\partial}{\partial x^0} := \frac{\partial}{\partial t} \equiv u$ then gives a basis $\{\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^{\bar A}}\}_{\bar A=1}^3 \equiv \{\frac{\partial}{\partial x^A}\}_{A=0}^3$ for the space-time tangent space, with coordinates $\{x^A\}_{A=0}^3$. From now on, it will be assumed that whenever a fluid source gauge is employed, the coordinates are arranged as just described, unless stated otherwise.

The following is a simple consequence of our choice of gauge and the fact that g is represented by the Minkovski metric in the frame formalism.

Lemma 3.2. *In fluid source gauge, it holds that*

$$e_0^A = \delta_0^A, \quad u^\alpha = \delta_0^\alpha, \quad \Gamma_{0\bar{\beta}}^{\bar{\alpha}} = 0,$$

$$\pi_{\alpha\beta} = g_{\alpha\beta} - \delta_{0\alpha}\delta_{0\beta}, \quad \pi_{\bar{\alpha}\bar{\beta}} = -\delta_{\bar{\alpha}\bar{\beta}}, \quad \pi_\alpha^\beta = \delta_\alpha^\beta - \delta_{0\alpha}\delta^{0\beta}.$$

3.2. *The reduced system of equations*

In this section, we investigate a reduced system for Eqs. (3.14). As mentioned in the introduction, its derivation, which follows that of [23, 24, 30], is given in Appendix B.

In light of the various symmetries involved, only equations for some components of the tensors involved are needed. First, let us determine them.

Taking the inner products of (3.3) with e_δ and using (3.4) gives

$$\Gamma_{\alpha\beta}^\beta|_{\not\Sigma} = 0, \tag{3.21}$$

$$\Gamma_{\bar{\alpha}\bar{\beta}}^0 = \Gamma_{\bar{\alpha}0}^{\bar{\beta}}, \tag{3.22}$$

$$\Gamma_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}} = -\Gamma_{\bar{\alpha}\bar{\gamma}}^{\bar{\beta}}, \tag{3.23}$$

where $(\cdot)|_{\not\Sigma}$ indicates that there is no sum over repeated indices, and from (3.4) and (3.21) we have

$$\Gamma_{00}^\alpha = \Gamma_{0\alpha}^0. \tag{3.24}$$

Identities (3.21)–(3.24) hold for any frame. Fixing the gauge allows further simplifications. From now on, we shall assume that our frame is a fluid source gauge, unless stated otherwise.

From Lemmas 3.1, 3.2 and the symmetries of the Weyl tensor,

$$E_{\alpha 0}(= E_{0\alpha}) = 0, \quad B_{\alpha 0}(= B_{0\alpha}) = 0, \tag{3.25}$$

$$E_{\bar{\alpha}\bar{\beta}} = W_{0\bar{\alpha}0\bar{\beta}}, \quad B_{\bar{\alpha}\bar{\beta}} = \frac{1}{2} W_{0\bar{\alpha}\mu\nu}\varepsilon_{0\bar{\beta}}^{\mu\nu}. \tag{3.26}$$

From (3.25) and Lemmas 3.1 and 3.2, it is seen that in fluid source gauge the Weyl tensor is determined from $E_{\bar{\alpha}\bar{\beta}}$ and $B_{\bar{\alpha}\bar{\beta}}$. It is therefore enough to consider equations for these quantities, whereas from Lemma 3.2 and identities (3.21)–(3.24), we obtain that it suffices to have evolution equations for connection coefficients $\Gamma_{\bar{\alpha}\bar{\gamma}}^{\bar{\beta}}$, $\Gamma_{0\bar{\alpha}}^0$, $\Gamma_{\bar{\alpha}\bar{\beta}}^0$ and frame coefficients $e_{\bar{\alpha}}^A$.

To write the reduced system, the introduction of yet another variable is needed. Write

$$s_\alpha = \nabla_\alpha s. \tag{3.27}$$

As was done for the quantities e_{β}^{A}, $\Gamma_{\alpha\beta}^{\gamma}$, $E_{\alpha\beta}$, $B_{\alpha\beta}$, we shall treat s_{α} as an unknown, with the relation (3.27) to be demonstrated after solutions are obtained.

Define the operator $\overline{\nabla}_{\mu}$ which acts on fields of the orthogonal complement of e_0 by

$$\overline{\nabla}_{\mu} A_{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_\ell} = \pi_{\mu}^{\nu}\nabla_{\nu} A_{\beta_1\beta_2\cdots\beta_\ell}\pi_{\bar{\alpha}_1}^{\beta_1}\pi_{\bar{\alpha}_2}^{\beta_2}\cdots\pi_{\bar{\alpha}_\ell}^{\beta_\ell}. \tag{3.28}$$

Then

$$\overline{\nabla}_{\mu}\pi_{\bar{\alpha}\bar{\beta}} = 0 \quad \text{and} \quad \overline{\nabla}_{\mu}\varepsilon_{\bar{\alpha}\bar{\beta}\bar{\gamma}} = 0.$$

Although, in fluid source gauge, it then follows that

$$\overline{\nabla}_{\bar{\mu}} A_{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_\ell} = \nabla_{\bar{\mu}} A_{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_\ell},$$

we shall explicitly write $\overline{\nabla}_{\bar{\mu}}$ to facilitate the comparison with Appendix B.

We can now investigate the reduced system, whose equations are

$$\partial_t e_{\bar{\beta}}^{A} + (\Gamma_{\bar{\beta}0}^{\bar{\mu}} - \Gamma_{0\bar{\beta}}^{\bar{\mu}})e_{\bar{\mu}}^{A} - \Gamma_{0\bar{\beta}}^{0}\delta_0^{A} = 0, \tag{3.29a}$$

$$\partial_t \Gamma_{\bar{\delta}\bar{\beta}}^{\bar{\alpha}} + \Gamma_{\bar{\lambda}\bar{\beta}}^{\bar{\alpha}}\Gamma_{\bar{\delta}0}^{\bar{\lambda}} + \Gamma_{00}^{\bar{\alpha}}\Gamma_{\bar{\delta}\bar{\beta}}^{0} - \Gamma_{\bar{\delta}0}^{\bar{\alpha}}\Gamma_{0\bar{\beta}}^{0} - \varepsilon_{\bar{\beta}}^{\mu\bar{\alpha}}B_{\bar{\delta}\mu} = 0, \tag{3.29b}$$

$$\begin{aligned}
&\partial_t \Gamma_{0\bar{\alpha}}^{0} - \nu^2 e_{\bar{\lambda}}(\Gamma_{\bar{\alpha}0}^{\bar{\lambda}}) + \Gamma_{0\bar{\mu}}^{0}\Gamma_{\bar{\alpha}0}^{\bar{\mu}} - \frac{\partial\nu^2}{\partial s}\Gamma_{\bar{\mu}0}^{\bar{\mu}}s_{\bar{\alpha}} + \frac{1}{p+\varrho}\\
&\quad\times\left(1 + \frac{r}{\nu^2}\frac{\partial\nu^2}{\partial r}\right)\frac{\partial p}{\partial s}\Gamma_{\bar{\mu}0}^{\bar{\mu}}s_{\bar{\alpha}} + \left(\frac{p+\varrho}{\nu^2}\left(\frac{\partial^2 p}{\partial\varrho^2}\right)_s - \nu^2\right)\Gamma_{\bar{\mu}0}^{\bar{\mu}}\Gamma_{0\bar{\alpha}}^{0}\\
&\quad - \nu^2[\Gamma_{\mu0}^{\bar{\lambda}}(\Gamma_{\bar{\alpha}\bar{\lambda}}^{\mu} - \Gamma_{\bar{\lambda}\bar{\alpha}}^{\mu}) - \Gamma_{\bar{\alpha}\mu}^{\bar{\lambda}}\Gamma_{\bar{\lambda}0}^{\mu} - \Gamma_{\bar{\lambda}\mu}^{\bar{\lambda}}\Gamma_{\bar{\alpha}0}^{\mu}]\\
&\quad - \frac{1}{p+\varrho}\nu^2\frac{\partial\varrho}{\partial s}\Gamma_{\bar{\mu}0}^{\bar{\mu}}s_{\bar{\alpha}} = 0,
\end{aligned} \tag{3.29c}$$

$$\begin{aligned}
&\nu^2\partial_t\Gamma_{\bar{\alpha}\bar{\beta}}^{0} - \nu^2 e_{\bar{\beta}}(\Gamma_{0\bar{\alpha}}^{0}) - \nu^2 E_{\bar{\alpha}\bar{\beta}} - \nu^2[\Gamma_{\mu\bar{\beta}}^{0}(\Gamma_{0\bar{\alpha}}^{\mu} - \Gamma_{\bar{\alpha}0}^{\bar{\mu}})\\
&\quad - \Gamma_{0\bar{\mu}}^{0}\Gamma_{\bar{\alpha}\bar{\beta}}^{\bar{\mu}} + \Gamma_{0\bar{\mu}}^{0}(\Gamma_{\bar{\alpha}\bar{\beta}}^{\bar{\mu}} - \Gamma_{\bar{\beta}\bar{\alpha}}^{\bar{\mu}}) + \nu^2\Gamma_{\bar{\mu}0}^{\bar{\mu}}(\Gamma_{\bar{\alpha}\bar{\beta}}^{0} - \Gamma_{\bar{\beta}\bar{\alpha}}^{0})\\
&\quad + \frac{1}{p+\varrho}\left(\frac{\partial\varrho}{\partial s} - \frac{1}{\nu^2}\frac{\partial p}{\partial s}\right)(\Gamma_{0\bar{\alpha}}^{0}s_{\bar{\beta}} - \Gamma_{0\bar{\beta}}^{0}s_{\bar{\alpha}})\\
&\quad - \frac{1}{6}\mathscr{K}(\varrho + 3p)\delta_{\bar{\alpha}\bar{\beta}}] = 0,
\end{aligned} \tag{3.29d}$$

$$\begin{aligned}
&\partial_t E_{\bar{\alpha}\bar{\beta}} + E_{\bar{\mu}\bar{\beta}}\Gamma_{\bar{\alpha}0}^{\bar{\mu}} + E_{\bar{\alpha}\bar{\mu}}\Gamma_{\bar{\beta}0}^{\bar{\mu}} + \overline{\nabla}_{\bar{\mu}}B_{\bar{\nu}(\bar{\alpha}}\varepsilon_{\bar{\beta})}^{\bar{\mu}\bar{\nu}} + 2\delta_{\bar{\mu}\bar{\lambda}}\Gamma_{00}^{\bar{\lambda}}\varepsilon_{(\bar{\alpha}}^{\bar{\mu}\bar{\nu}}B_{\bar{\beta})\bar{\nu}}\\
&\quad - 3\delta^{\bar{\mu}\bar{\lambda}}\Gamma_{(\bar{\alpha}|\bar{\lambda}|}^{0}E_{\bar{\beta})\bar{\mu}} - 2\delta^{\bar{\mu}\bar{\lambda}}\Gamma_{\bar{\lambda}(\bar{\alpha}}^{0}E_{\bar{\beta})\bar{\mu}} + \delta_{\bar{\alpha}\bar{\beta}}\delta^{\bar{\mu}\bar{\lambda}}\delta^{\bar{\nu}\bar{\sigma}}\Gamma_{\bar{\lambda}\bar{\sigma}}^{0}E_{\bar{\mu}\bar{\nu}}\\
&\quad + 2\delta^{\bar{\mu}\bar{\nu}}\Gamma_{\bar{\mu}\bar{\nu}}^{0}E_{\bar{\alpha}\bar{\beta}} - \frac{1}{4}\mathscr{K}(p+\varrho)\left(\Gamma_{\bar{\alpha}\bar{\beta}}^{0} + \Gamma_{\bar{\beta}\bar{\alpha}}^{0}\right.\\
&\quad\left. - \frac{2}{3}\delta_{\bar{\alpha}\bar{\beta}}\delta^{\bar{\mu}\bar{\nu}}\Gamma_{\bar{\mu}\bar{\nu}}^{0}\right) = 0,
\end{aligned} \tag{3.29e}$$

$$\partial_t B_{\bar{\alpha}\bar{\beta}} + B_{\bar{\mu}\bar{\beta}} \Gamma^{\bar{\mu}}_{\bar{\alpha}0} + B_{\bar{\alpha}\bar{\mu}} \Gamma^{\bar{\mu}}_{\bar{\beta}0} - \overline{\nabla}_{\bar{\mu}} E_{\bar{\nu}(\bar{\alpha}} \varepsilon^{\bar{\mu}\bar{\nu}}_{\bar{\beta})} - 2\delta_{\bar{\mu}\bar{\lambda}} \Gamma^{\bar{\lambda}}_{00} \varepsilon^{\bar{\mu}\bar{\nu}}_{(\bar{\alpha}} E_{\bar{\beta})\bar{\nu}}$$
$$- \delta^{\bar{\mu}\bar{\lambda}} \Gamma^0_{\bar{\lambda}(\bar{\alpha}} B_{\bar{\beta})\bar{\mu}} - 2\delta^{\bar{\mu}\bar{\lambda}} \Gamma^0_{(\bar{\alpha}|\bar{\lambda}|} B_{\bar{\beta})\bar{\mu}} + \delta^{\bar{\mu}\bar{\nu}} \Gamma^0_{\bar{\mu}\bar{\nu}} B_{\bar{\alpha}\bar{\beta}}$$
$$- \Gamma^0_{\bar{\mu}\bar{\nu}} B_{\bar{\sigma}\bar{\lambda}} \varepsilon^{\bar{\sigma}\bar{\mu}}_{(\bar{\alpha}} \varepsilon^{\bar{\nu}\bar{\lambda}}_{\bar{\beta})} = 0, \tag{3.29f}$$

$$\partial_t \varrho + (p + \varrho)\Gamma^{\mu}_{\mu 0} = 0, \tag{3.29g}$$

$$\partial_t s = 0, \tag{3.29h}$$

$$\partial_t s_\alpha - (\Gamma^{\mu}_{0\alpha} - \Gamma^{\mu}_{\alpha 0}) s_\mu = 0, \tag{3.29i}$$

$$\partial_t r + r\Gamma^{\mu}_{\mu 0} = 0, \tag{3.29j}$$

where it is understood that in the expressions involving partial derivatives of ϱ with respect to the matter variables, ϱ is to be replaced by $\mathcal{P}$ (since (2.11) is not a part of the above system); p and ν^2 are given by (2.12) and (2.22), respectively; and $(\cdot)_{[\alpha|\mu|\beta]}$ means anti-symmetrization of the indices α and β only. The unknowns to be determined are $e^A_{\bar{\alpha}}, \Gamma^{\bar{\beta}}_{\bar{\alpha}\bar{\gamma}}, \Gamma^0_{0\bar{\alpha}}, \Gamma^0_{\bar{\alpha}\bar{\gamma}}, E_{\bar{\alpha}\bar{\beta}}, B_{\bar{\alpha}\bar{\beta}}, \varrho, r, s, s_\alpha$. It is explicitly assumed that E and B are symmetric, therefore Eqs. (3.29e) and (3.29f) are written only for the independent components of these fields, say $\bar{\alpha} \leq \bar{\beta}$, with the remaining components defined by these symmetry relations. Similarly, all the symmetries (3.21)–(3.24) and the gauge condition $\Gamma^{\bar{\beta}}_{0\bar{\alpha}} = 0$ from Lemma 3.2 are assumed, with Eqs. (3.29b)–(3.29d) written only for the independent components and the remaining ones being defined by these symmetry relations. We write the unknowns collectively as a vector

$$z = (e^A_{\bar{\alpha}}, \Gamma^{\bar{\beta}}_{\bar{\alpha}\bar{\gamma}}, \Gamma^0_{0\bar{\alpha}}, \Gamma^0_{\bar{\alpha}\bar{\gamma}}, E_{\bar{\alpha}\bar{\beta}}, B_{\bar{\alpha}\bar{\beta}}, \varrho, r, s, s_\alpha). \tag{3.30}$$

Remark 3.3. In order that some components of Γ, E and B be defined by symmetry, as mentioned above, it is necessary that the initial conditions also obey such relations. Given arbitrary $E_{\bar{\alpha}\bar{\beta}}|_{t=0}$, $1 \leq \bar{\alpha} \leq \bar{\beta} \leq 3$, one can in principle always define the remaining $E_{\bar{\alpha}\bar{\beta}}|_{t=0}$ by $E_{\bar{\alpha}\bar{\beta}} = E_{\bar{\beta}\bar{\alpha}}$. However, in the case of interest, E has to be determined from an initial data set, in which case all its components will be given on the initial slice (see Proposition 4.1 below). In this situation, the symmetry of E at $t = 0$ has to be demonstrated — and only then is one allowed to the write system (3.29) solely for $\{E_{\bar{\alpha}\bar{\beta}}\}_{\bar{\alpha}\leq\bar{\beta}}$ and impose symmetry relations for the remaining components. Similar statements hold for the other fields involving symmetries.

4. Proof of Theorem 2.1

4.1. *Initial data*

In order to address the solvability of the systems (3.29), we need to provide suitable initial conditions. These should be determined entirely by the initial data for the Einstein–Euler-Entropy system (as this is the set of equations we are ultimately interested in) and our gauge choices. We also have to show that the initial data for E and B, which are naturally constructed from the initial data set, are indeed symmetric and trace-free. Although all of this can be inferred from similar works treating the vacuum and conformal vacuum Einstein equations, as well as their coupling to the Yang–Mills equations [25–29], an explicit proof does not seem to be available in the literature in the case of our system with our gauge choices. It is therefore useful to provide it here.

Proposition 4.1. *Let* $\mathcal{I} = (\Sigma, g_0, \kappa, r_0, s_0, v, \mathscr{P})$ *be an initial data set for the Einstein–Euler-Entropy system, with Einsteinian development* $(\mathcal{M}, g)$ *that is a perfect fluid source where the Einstein–Euler-Entropy system of equations is satisfied. Let* $\{e_\alpha\}_{\alpha=0}^3$ *be a fluid source gauge defined on a coordinate chart* U *of* $\mathcal{M}$*, and let* z *be as in* (3.30). *Then*

$$z|_{\Sigma\cap U}$$

can be written in terms of quantities determined entirely by $\mathcal{I}$. *Furthermore,* $E|_\Sigma$ *and* $B|_\Sigma$ *are symmetric and trace-free.*

Proof. It will be useful to first express space-time quantities in terms of an adapted frame. Let $\{\widetilde{e}_{\bar{\alpha}}\}_{\bar{\alpha}=1}^3$ be a frame on Σ orthonormal with respect to g_0. Denote by $\widetilde{e}_0$ the future directed unit normal (with respect to g) of Σ. The frame $\{\widetilde{e}_\alpha\}_{\alpha=0}^3$ is extended to $\mathcal{M}$ by parallel transport in the direction of $\widetilde{e}_0$. Let $\{\widetilde{x}^{\bar{A}}\}_{\bar{A}=1}^3$ be coordinates on Σ. These coordinates are extended to $\mathcal{M}$ by dragging them along $\widetilde{e}_0$, so that they are constant on the integral curves of $\widetilde{e}_0$. Denoting by $\widetilde{x}^0$ the parameter of such curves, we obtain that $\{\widetilde{x}^A\}_{A=0}^3$ is a coordinate system on $\mathcal{M}$. The frame and coordinate basis are related by

$$\widetilde{e}_\alpha = \widetilde{e}_\alpha^A \frac{\partial}{\partial \widetilde{x}^A}.$$

By construction, it holds that

$$\widetilde{e}_0^A = \delta_0^A, \quad \widetilde{e}_{\bar{\alpha}}^0 = 0.$$

Quantities expressed in terms of $\widetilde{e}_\alpha$ and $\widetilde{x}^A$ will be denoted with a tilde $\widetilde{\ }$, and the same index convention as of Notation 2 is assumed. In particular, in the frame $\widetilde{e}_\alpha$, the metric g_0 is written as

$$\widetilde{g}_{0\bar{\alpha}\bar{\beta}} = \mathrm{diag}(-1, -1, -1),$$

and the connection coefficients are given by

$$\nabla_{\widetilde{e}_\alpha} \widetilde{e}_\beta = \widetilde{\Gamma}_{\alpha\beta}^\gamma \widetilde{e}_\gamma.$$

Notice that the space-time metric is represented by the (constant) matrix $\mathrm{diag}(1, -1, -1, -1)$ in both frames e_α and $\widetilde{e}_\alpha$. This allows us to drop the $\widetilde{\ }$ from the metric $\widetilde{g}_{\alpha\beta}$, but we still write $\widetilde{g}_{\alpha\beta}$ when we want to stress that some expression is written in the $\widetilde{e}_\alpha$ basis. (The distinction has to be maintained though when g is written in the coordinate basis $\{\frac{\partial}{\partial \widetilde{x}^A}\}_{A=0}^3$ and $\{\frac{\partial}{\partial x^A}\}_{A=0}^3$.)

Arguing similarly to (3.21)–(3.24),

$$\begin{aligned} \widetilde{\Gamma}^{\gamma}_{0\beta} = \widetilde{\Gamma}^{0}_{\bar{\alpha}0} = \widetilde{\Gamma}^{\gamma}_{0\bar{\beta}} = 0, \\ \widetilde{\Gamma}^{0}_{\bar{\alpha}\bar{\beta}} = \widetilde{\Gamma}^{\bar{\beta}}_{\bar{\alpha}0} = -\widetilde{\kappa}_{\bar{\alpha}\bar{\beta}}. \end{aligned} \tag{4.1}$$

Now we proceed to relate the tilded quantities to z. Without loss of generality, it can be assumed that[s]

$$\frac{\partial}{\partial x^{\bar{A}}} = \frac{\partial}{\partial \widetilde{x}^{\bar{A}}} \quad \text{on } \Sigma. \tag{4.2}$$

Write

$$e_{\bar{\alpha}} = e^{0}_{\bar{\alpha}} \frac{\partial}{\partial x^0} + e^{\bar{A}}_{\bar{\alpha}} \frac{\partial}{\partial x^{\bar{A}}}.$$

On Σ, $e^{\bar{A}}_{\bar{\alpha}} \frac{\partial}{\partial x^{\bar{A}}}$ can be written in terms of $\widetilde{e}_{\bar{\alpha}}$, which in turn is determined by g_0. As for $e^0_{\bar{\alpha}}$, compute

$$\begin{aligned} \langle e_{\bar{\alpha}}, e_{\bar{\alpha}} \rangle = -1 &= \left\langle e^{A}_{\bar{\alpha}} \frac{\partial}{\partial x^{A}}, e^{B}_{\bar{\alpha}} \frac{\partial}{\partial x^{B}} \right\rangle \\ &= (e^0_{\bar{\alpha}})^2 + 2 e^{\bar{A}}_{\bar{\alpha}} e^{0}_{\bar{\alpha}} g_{\bar{A}0} + e^{\bar{A}}_{\bar{\alpha}} e^{\bar{B}}_{\bar{\alpha}} g_{\bar{A}\bar{B}}, \end{aligned} \tag{4.3}$$

where we have used that $g_{00} = \langle e_0, e_0 \rangle \equiv \langle \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^0} \rangle = 1$. But since the projection of u onto Σ is v, it holds that

$$v_{\bar{\alpha}} = \langle u, \widetilde{e}_{\bar{\alpha}} \rangle = \left\langle u^B \frac{\partial}{\partial x^B}, \widetilde{e}^{\bar{A}}_{\bar{\alpha}} \frac{\partial}{\partial x^{\bar{A}}} \right\rangle = \widetilde{e}^{\bar{A}}_{\bar{\alpha}} g_{0\bar{A}} \quad \text{on } \Sigma, \tag{4.4}$$

where $u^A = \delta^A_0$ and (4.2) have been used. From (4.4), we can now get $g_{0\bar{A}}$ on Σ. Therefore, restricting (4.3) to Σ allows us to solve for for $e^0_{\bar{\alpha}}|_\Sigma$ since $e_{\bar{\alpha}}$ is space-like and unit, with the sign of $e^0_{\bar{\alpha}}$ being unambiguously defined by our choice of orientation. Therefore, $e^0_{\bar{\alpha}}|_\Sigma$ is also determined by the initial data. Because $e^A_0 = \delta^A_0$, we conclude that all the functions e^A_α are determined on Σ by the initial data (and of course our gauge choice), the same being true for g_{AB} in light of (4.4) and $g_{00} = 1$.

The change of basis from $\{\widetilde{e}_\alpha\}$ to $\{e_\alpha\}$ is given by a Lorentz transformation Λ:

$$e_\alpha = \Lambda^{\mu}_{\alpha} \widetilde{e}_\mu. \tag{4.5}$$

Taking the inner product of (4.5) with $\frac{\partial}{\partial x^{\bar{A}}}$ and restricting to Σ produces

$$e^{B}_{\alpha} g_{B\bar{A}} = \Lambda^{\bar{\mu}}_{\alpha} \widetilde{e}^{\bar{B}}_{\bar{\mu}} \widetilde{g}_{\bar{B}\bar{A}} \quad \text{on } \Sigma, \tag{4.6}$$

where $\langle \widetilde{e}_0, \frac{\partial}{\partial \widetilde{x}^{\bar{A}}} \rangle = 0$ has been used. From our previous relations, it follows that all quantities on (4.6), except possibly the Λ's themselves, are determined by the initial data on Σ. Viewing (4.6) as a system for the $\Lambda^{\bar{\mu}}_{\alpha}$, ($\alpha = 0, \ldots, 3$, $\bar{\mu} = 1, 2, 3$) (which will be invertible since the matrix of the system is that of the change from

[s]Then on Σ it also holds that $g_{\bar{A}\bar{B}} = \widetilde{g}_{\bar{A}\bar{B}}$, but it will be convenient to write the ˜ in the metric as a way to keep track of the changes of basis.

$\frac{\partial}{\partial \bar{x}^{\bar{A}}}$ to $\widetilde{e}_{\bar{\mu}}$ up to a lowering of the indices) shows that $\Lambda^{\bar{\mu}}_{\alpha}|_{\Sigma}$ is entirely determined by the initial data and the gauge choices. Recalling the identity

$$\Lambda^{\alpha}_{\gamma}\Lambda^{\beta}_{\delta} = g_{\gamma\delta} \tag{4.7}$$

then gives

$$(\Lambda^0_0)^2 = 1 + \sum_{\bar{\mu}=1}^{3}(\Lambda^{\bar{\mu}}_0)^2,$$

so that Λ^0_0 is also determined only by the initial data when restricted to Σ. Notice that the sign of Λ^0_0 is positive because Λ belongs to the proper Lorentz group.[t] From (4.7), the remaining Λ's are given in terms of $\Lambda^{\bar{\mu}}_{\alpha}$ and we conclude that $\Lambda|_{\Sigma}$ is determined by the initial data.

Next we investigate the connection coefficients. Taking the inner products of (3.3) with e_δ,

$$g_{\delta\gamma}\Gamma^{\gamma}_{\alpha\beta} = \langle \nabla_{\alpha} e_{\beta}, e_{\delta}\rangle. \tag{4.8}$$

Writing the frames on the right-hand side of the above expression in terms of $\widetilde{e}_{\alpha}$ via (4.5) leads to

$$\Gamma^{\xi}_{\alpha\beta} = g_{\nu\sigma}\Lambda^{\sigma}_{\gamma}g^{\gamma\xi}\Lambda^{\mu}_{\alpha}\widetilde{e}_{\mu}(\Lambda^{\nu}_{\beta}) + g_{\tau\sigma}\Lambda^{\sigma}_{\gamma}g^{\gamma\xi}\Lambda^{\nu}_{\beta}\Lambda^{\mu}_{\alpha}\widetilde{\Gamma}^{\tau}_{\mu\nu}. \tag{4.9}$$

On Σ, the coefficients $\widetilde{\Gamma}^{\bar{\tau}}_{\bar{\mu}\bar{\nu}}$ are determined by g_0. Then, by virtue of (4.1), the construction of $\widetilde{e}_0$ and $\widetilde{e}_{\bar{\mu}}$, and our previous relations involving Λ, it follows that all terms on the right-hand side of (4.9) are determined by the initial data, except possibly those involving the derivatives of the Lorentz transformation in the direction of $\widetilde{e}_0$. To see how such terms are determined, recall that in fluid source gauge $\Gamma^{\bar{\xi}}_{0\bar{\beta}} = 0$, so that (4.9) gives

$$\begin{aligned} 0 = {} & \Lambda^0_{\gamma}g^{\gamma\bar{\xi}}\Lambda^0_0\widetilde{e}_0(\Lambda^0_{\bar{\beta}}) + g_{\bar{\nu}\sigma}\Lambda^{\sigma}_{\gamma}g^{\gamma\bar{\xi}}\Lambda^{\bar{\mu}}_0\widetilde{e}_{\bar{\mu}}(\Lambda^{\bar{\nu}}_{\bar{\beta}}) \\ & + g_{\tau\sigma}\Lambda^{\sigma}_{\gamma}g^{\gamma\bar{\xi}}\Lambda^{\nu}_{\bar{\beta}}\Lambda^{\mu}_0\widetilde{\Gamma}^{\tau}_{\mu\nu}. \end{aligned}$$

Because $\Lambda^0_0 \geq 1$, we see that

$$\Lambda^0_{\gamma}g^{\gamma\bar{\xi}}\widetilde{e}_0(\Lambda^0_{\bar{\beta}})|_{\Sigma} \tag{4.10}$$

can be written in terms of quantities determined by $\mathcal{I}$. Using then (4.10) into (4.9) with $\alpha \mapsto \bar{\alpha}, \beta \mapsto \bar{\beta}$ and $\xi \mapsto \bar{\xi}$ shows that $\Gamma^{\bar{\xi}}_{\bar{\alpha}\bar{\beta}}|_{\Sigma}$ is also written solely in terms of quantities coming from $\mathcal{I}$. The coefficients $\Gamma^0_{0\bar{\beta}}|_{\Sigma}$ and $\Gamma^0_{\bar{\alpha}\bar{\beta}}|_{\Sigma}$ are similarly determined: from $\Gamma^0_{00} = 0$ and (4.9) one obtains that $\widetilde{e}_0(\Lambda^0_0)|_{\Sigma}$ is determined by $\mathcal{I}$; using this fact into the expressions for $\Gamma^{\bar{\xi}}_{00}$ and $\Gamma^{\bar{\xi}}_{\bar{\alpha}0}$ shows that the same is true for these quantities restricted to Σ, and hence the claim follows upon evoking (3.22) and (3.24).

[t]Since u belongs to the inside of the future light-cone.

The restrictions of ϱ, s and r to Σ have the desired properties in that they are just scalar functions. For s_α, our gauge choice and (2.15) imply $s_0 \equiv 0$. Then, recalling that $\frac{\partial}{\partial x^0} = e_0$,

$$s_{\bar{\alpha}} = \nabla_{e_{\bar{\alpha}}} s = e^{\bar{A}}_{\bar{\alpha}} \frac{\partial s}{\partial x^{\bar{A}}},$$

which is determined by $\mathcal{I}$ on Σ by the above results for $e^{\bar{A}}_{\bar{\alpha}}$ and noticing that $\frac{\partial s}{\partial x^{\bar{A}}}|_\Sigma = \frac{\partial \varsigma_0}{\partial x^{\bar{A}}}$.

From Definition 3.1,

$$\widetilde{E}_{\alpha\beta} = \widetilde{W}_{0\alpha 0\beta}\widetilde{u}^0\widetilde{u}^0 + \widetilde{W}_{0\alpha\bar{\sigma}\beta}\widetilde{u}^0\widetilde{u}^{\bar{\sigma}} + \widetilde{W}_{\bar{\mu}\alpha 0\beta}\widetilde{u}^{\bar{\mu}}u^0 + \widetilde{W}_{\bar{\mu}\alpha\bar{\sigma}\beta}\widetilde{u}^{\bar{\mu}}\widetilde{u}^{\bar{\sigma}}. \tag{4.11}$$

On Σ we have $\widetilde{u}^{\bar{\mu}} = \widetilde{v}^{\bar{\mu}}$, while $\widetilde{u}^0|_\Sigma$ is computed from the $\widetilde{v}$ and the normalization condition $u^\alpha u_\alpha = 1$. Therefore, to show that $\widetilde{E}_{\alpha\beta}|_\Sigma$ is written in terms of quantities determined by $\mathcal{I}$, we only need to investigate the components of the Weyl tensor in (4.11), and by its symmetries, it suffices to do so for components of the form $\widetilde{W}_{0\bar{\beta}0\bar{\delta}}$, $\widetilde{W}_{0\bar{\alpha}\bar{\sigma}\bar{\delta}}$ and $\widetilde{W}_{\bar{\mu}\bar{\alpha}\bar{\sigma}\bar{\beta}}$.

From the Gauss equation, the decomposition of the Riemann tensor (i.e. (3.5) with $d_{\alpha\beta\gamma\delta} \equiv 0$), Einstein equations, and (3.12), we obtain

$$\begin{aligned}\widetilde{W}_{0\bar{\beta}0\bar{\delta}} &= \frac{1}{2}\mathscr{K}\widetilde{T}_{\bar{\beta}\bar{\delta}} - \frac{1}{6}\mathscr{K}\widetilde{T}\,\widetilde{g}_{\bar{\beta}\bar{\delta}} - \frac{1}{2}\mathscr{K}\widetilde{T}_{00}\widetilde{g}_{\bar{\beta}\bar{\delta}} \\ &\quad - {}^{(3)}\widetilde{R}_{\bar{\beta}\bar{\delta}} - \widetilde{\kappa}^{\bar{\lambda}}_{\bar{\lambda}}\widetilde{\kappa}_{\bar{\beta}\bar{\delta}} + \widetilde{\kappa}^{\bar{\lambda}}_{\bar{\delta}}\widetilde{\kappa}_{\bar{\beta}\bar{\lambda}} \text{ on } \Sigma,\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}\widetilde{W}_{\bar{\mu}\bar{\alpha}\bar{\sigma}\bar{\beta}} &= {}^{(3)}\widetilde{R}_{\bar{\mu}\bar{\alpha}\bar{\sigma}\bar{\beta}} + \widetilde{\kappa}_{\bar{\mu}\bar{\sigma}}\widetilde{\kappa}_{\bar{\alpha}\bar{\beta}} - \widetilde{\kappa}_{\bar{\mu}\bar{\beta}}\widetilde{\kappa}_{\bar{\alpha}\bar{\sigma}} - \frac{1}{2}\mathscr{K}\left(\widetilde{T}_{\bar{\beta}\bar{\alpha}} - \frac{1}{3}\widetilde{T}\,\widetilde{g}_{\bar{\beta}\bar{\alpha}}\right)\widetilde{g}_{\bar{\mu}\bar{\sigma}} \\ &\quad + \frac{1}{2}\mathscr{K}\left(\widetilde{T}_{\bar{\sigma}\bar{\alpha}} - \frac{1}{3}\widetilde{T}\,\widetilde{g}_{\bar{\sigma}\bar{\alpha}}\right)\widetilde{g}_{\bar{\mu}\bar{\beta}} + \frac{1}{2}\mathscr{K}\left(\widetilde{T}_{\bar{\beta}\bar{\mu}} - \frac{1}{3}\widetilde{T}\widetilde{g}_{\bar{\beta}\bar{\mu}}\right)\widetilde{g}_{\bar{\alpha}\bar{\sigma}} \\ &\quad - \frac{1}{2}\mathscr{K}\left(\widetilde{T}_{\bar{\sigma}\bar{\mu}} - \frac{1}{3}\widetilde{T}\,\widetilde{g}_{\bar{\sigma}\bar{\mu}}\right)\widetilde{g}_{\bar{\alpha}\bar{\beta}} \text{ on } \Sigma,\end{aligned} \tag{4.13}$$

where ${}^{(3)}\widetilde{R}_{\bar{\beta}\bar{\delta}}$ and ${}^{(3)}\widetilde{R}_{\bar{\mu}\bar{\alpha}\bar{\sigma}\bar{\beta}}$ are, respectively, the Ricci and Riemann curvatures of (Σ, g_0). In a similar fashion but using now the Codazzi equation:

$$\widetilde{W}_{0\bar{\alpha}\bar{\sigma}\bar{\beta}} = {}^{(3)}\nabla_{\bar{\sigma}}\widetilde{\kappa}_{\bar{\beta}\bar{\alpha}} - {}^{(3)}\nabla_{\bar{\beta}}\widetilde{\kappa}_{\bar{\sigma}\bar{\alpha}} + \frac{1}{2}\mathscr{K}\widetilde{T}_{0\bar{\beta}}\widetilde{g}_{\bar{\alpha}\bar{\sigma}} - \frac{1}{2}\mathscr{K}\widetilde{T}_{0\bar{\sigma}}\widetilde{g}_{\bar{\alpha}\bar{\beta}} \quad \text{on } \Sigma, \tag{4.14}$$

where ${}^{(3)}\nabla$ is Levi-Civita connection of g_0.

From (2.3), (2.11), (2.12), (4.11)–(4.14), we obtain that $\widetilde{E}_{\alpha\beta}|_\Sigma$ is a tensor solely determined by $\mathcal{I}$, which is symmetric and trace-free by the constraint equations (2.16) and (2.17). The same holds for $E_{\alpha\beta}|_\Sigma$ by the invariance of the trace and the properties of Λ^μ_ν previously shown. By an analogous argument, a similar statement holds for $B_{\alpha\beta}|_\Sigma$. □

Definition 4.1. By Proposition 4.1, given an initial data set, a choice of (fluid source) gauge uniquely determines initial conditions for the reduced system. These initial conditions for the reduced system are henceforth called a *reduced initial data set.*

For practical applications, e.g., to numerically solve the equations, Proposition 4.1 tells us how to arrange the initial data. Given a negative three-dimensional Riemannian manifold (Σ, g_0), choose coordinates $\{\frac{\partial}{\partial \widetilde{x}^{\bar{A}}}\}_{\bar{A}=1}^3$ and an orthonormal frame $\{\widetilde{e}_{\bar{\alpha}}\}_{\bar{\alpha}=1}^3$. Declare a metric on $[0,T] \times \Sigma$ by $g = d\widetilde{t}^{\,2} + g_0$, where the tilde emphasizes that we identify $[0,T]$ with the time coordinate $\widetilde{x}^0$ (so that $\widetilde{e}_0 = \frac{\partial}{\partial \widetilde{x}^0}$), and the metric is written in terms of the resulting coordinates on $[0,T] \times \Sigma$. Using this metric and the (given) three-velocity v we determine the four velocity u on Σ by the condition $\langle u, u\rangle = 1$ (notice that flowing along u produces the parameter t, i.e. x_0). Besides, we complete u to an orthonormal frame $\{e_\alpha\}_{\alpha=0}^3$ at the instant $\widetilde{x}^0 = 0$. The relations given in Proposition 4.1 can now be used to *define* the remaining quantities such as $\Lambda_0^0, \widetilde{\Gamma}^{\xi}_{\ \alpha\beta}, \Gamma^{\xi}_{\ \alpha\beta}$, etc., on the initial Cauchy surface.

4.2. *Well-posedness of the reduced system*

Short-time existence for the reduced system is a direct consequence of the way it has been set up, at least under our hypotheses. In fact, the gauge choice and construction of (3.29), originally devised by Friedrich [23], are motivated exactly by the attempt of obtaining a reduced system that is symmetric hyperbolic, in which case well-known results can be applied. There are, however, one subtlety and one observation, that have to be dealt with. First, the initial conditions for the $e^A_{\bar{\alpha}}$, Γ, E and B involve a different number of derivatives of the initial metric g_0, hence they belong to $H^\ell_{ul}(\Sigma)$ with different values of ℓ. The usual techniques of symmetric hyperbolic systems, however, yield solutions with the same regularity for all the unknowns (see, e.g., [41, 58]), which in this case would be that of the less regular initial data, namely, E and B. This does not give the desired differentiability for the frame coefficients, nor for the metric. Second, although at first glance the matrix coefficient of $\frac{\partial}{\partial t}$ appears to be a diagonal matrix with entries either 1 or ν^2, the "spatial" derivatives $e_{\bar{\mu}}$ involve derivatives in the direction of x^0 hence contributing to the zeroth matrix coefficient.

Proposition 4.2. *Let $\mathcal{I}$ be an initial data set for the Einstein–Euler-Entropy system satisfying the hypotheses of Theorem* 2.1. *Fix a positive real number T_E and consider the system* (3.29) *defined on $[0,T_E] \times \Sigma$, where x^0 is identified with the time coordinate on $[0,T_E]$. Let $\mathcal{I}_0$ be the reduced initial data set determined on $\{0\} \times \Sigma$ by $\mathcal{I}$. Then there exists a unique solution z to* (3.29) *on some time interval $[0,T_E'], 0 < T_E' \leq T_E$, and satisfying $z|_{t=0} = \mathcal{I}_0$. Furthermore, $e^{\bar{A}}_{\bar{\alpha}} \in C^0([0,T_E'], H^{s+1}_{ul}(\Sigma)) \cap C^1([0,T_E'], H^s_{ul}(\Sigma)), e^0_{\bar{\alpha}} \in C^0([0,T_E'], H^s_{ul}(\Sigma)) \cap C^1([0,T_E'], H^{s-1}_{ul}(\Sigma)), \Gamma^{\bar{\beta}}_{\bar{\alpha}\bar{\gamma}}, \Gamma^0_{0\bar{\alpha}}, \Gamma^0_{\bar{\alpha}\bar{\gamma}} \in C^0([0,T_E'], H^s_{ul}(\Sigma)) \cap C^1([0,T_E'], H^{s-1}_{ul}(\Sigma)), E_{\bar{\alpha}\bar{\beta}}, B_{\bar{\alpha}\bar{\beta}} \in C^0([0,T_E'], H^{s-1}_{ul}(\Sigma)) \cap C^1([0,T_E'], H^{s-2}_{ul}(\Sigma)), s_\alpha \in C^0([0,T_E'], H^{s-1}_{ul}(\Sigma)) \cap C^1([0,T_E'], H^{s-2}_{ul}(\Sigma)), \varrho, r, s \in C^0([0,T_E'], H^s_{ul}(\Sigma)) \cap C^1([0,T_E'], H^{s-1}_{ul}(\Sigma))$.*

Proof. First, we claim that (3.29) is symmetric hyperbolic with respect to t on the initial hypersurface and remains so as long as $\nu^2 > 0$, and the slices $\Sigma_t = \{t = \text{constant}\}$ are space-like with respect to the quadratic form g_t induced by the frame coefficients. Notice that symmetry here means symmetry of the matrix coefficients M^A of the derivatives $\frac{\partial}{\partial x^A}$. Therefore we have to first change basis via (3.1). Other than the first term in each equation, the derivative $\frac{\partial}{\partial t}$ also figures in the terms involving $e_{\bar{\mu}}$ in equations (3.29c), (3.29d), (3.29e) and (3.29f), where in these last two equations the contribution of $e_{\bar{\mu}}$ comes from the covariant derivatives. Expressing all derivatives in (3.29) in terms of $\frac{\partial}{\partial x^A}$ gives that the term in ∂_t can be written symbolically as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\nu^2 e^0_{\bar{\beta}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\nu^2 e^0_{\bar{\beta}} & \nu^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m^t_{\bar{\alpha}\bar{\beta}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} e^A_{\bar{\beta}} \\ \Gamma^{\bar{\alpha}}_{\bar{\delta}\bar{\beta}} \\ \Gamma^0_{0\bar{\alpha}} \\ \Gamma^0_{\bar{\alpha}\bar{\beta}} \\ (E_{\bar{\alpha}\bar{\beta}},\, B_{\bar{\alpha}\bar{\beta}}) \\ \varrho \\ s \\ s_\alpha \\ r \end{pmatrix} \tag{4.15}$$

where $m^t_{\bar{\alpha}\bar{\beta}}$ is the matrix part corresponding to

$$\begin{cases} \partial_t E_{\bar{\alpha}\bar{\beta}} + \dfrac{1}{2} e^0_{\bar{\mu}} \varepsilon^{\bar{\mu}\bar{\nu}}_{\bar{\beta}} \partial_t B_{\bar{\nu}\bar{\alpha}} + \dfrac{1}{2} e^0_{\bar{\mu}} \varepsilon^{\bar{\mu}\bar{\nu}}_{\bar{\alpha}} \partial_t B_{\bar{\nu}\bar{\beta}}, \\ \partial_t B_{\bar{\alpha}\bar{\beta}} - \dfrac{1}{2} e^0_{\bar{\mu}} \varepsilon^{\bar{\mu}\bar{\nu}}_{\bar{\beta}} \partial_t E_{\bar{\nu}\bar{\alpha}} - \dfrac{1}{2} e^0_{\bar{\mu}} \varepsilon^{\bar{\mu}\bar{\nu}}_{\bar{\alpha}} \partial_t E_{\bar{\nu}\bar{\beta}}. \end{cases} \tag{4.16}$$

From (4.15) and (4.16), it is seen that $M^0 \equiv M^t$ is symmetric. Symmetry of the remaining $M^{\bar{A}}, \bar{A} = 1, 2, 3$, is similarly verified.

The quadratic form g_t is given by

$$g_{t\,\bar{A}\bar{B}} = f^0_{\bar{A}} f^0_{\bar{B}} - \sum_{\bar{\alpha}=1}^{3} f^{\bar{\alpha}}_{\bar{A}} f^{\bar{\alpha}}_{\bar{B}}, \tag{4.17}$$

where the coefficients $f^\alpha_{\bar{A}}$ are defined via

$$\frac{\partial}{\partial x^{\bar{A}}} = f^{\bar{\alpha}}_{\bar{A}} e_{\bar{\alpha}} + f^0_{\bar{A}} e_0 \equiv f^{\bar{\alpha}}_{\bar{A}} e_{\bar{\alpha}} + f^0_{\bar{A}} \frac{\partial}{\partial t}.$$

From these constructions, we obtain that the characteristics of the system are non-zero multiples of

$$\xi_0^{K_1} \left(\xi_0^2 + \frac{1}{4} \pi^{\bar{\alpha}\bar{\beta}} \xi_{\bar{\alpha}\bar{\beta}} \right)^{K_2} (\xi_0^2 + \nu^2 \pi^{\bar{\gamma}\bar{\delta}} \xi_{\bar{\gamma}\bar{\delta}})^{K_3} (g^{\lambda\tau} \xi_\lambda \xi_\tau)^{K_4},$$

where $K_1, \dots, K_4$ are positive integers. It follows that the system is symmetric hyperbolic as long as $\nu^2 > 0$ and g_t remains negative definite.

Consider now the problem on a local patch $[0, T_E] \times U$. From our hypotheses and the constructions of Proposition 4.1, it follows that the initial data $\mathcal{I}_0 \equiv z(0, \cdot)$ is such that

$$e_{\bar{\alpha}}^0(0, \cdot) \in H^s(U), e_{\bar{\alpha}}^{\bar{A}}(0, \cdot) \in H^{s+1}(U), \tag{4.18}$$

$$(\varrho, r, s)(0, \cdot) \in H^s(U), \tag{4.19}$$

$$(\Gamma_{\bar{\alpha}\bar{\gamma}}^{\bar{\beta}}, \Gamma_{0\bar{\alpha}}^0, \Gamma_{\bar{\alpha}\bar{\gamma}}^0)(0, \cdot) \in H^s(U), \tag{4.20}$$

$$(E_{\bar{\alpha}\bar{\beta}}, B_{\bar{\alpha}\bar{\beta}}, s_\alpha)(0, \cdot) \in H^{s-1}(U). \tag{4.21}$$

$e_{\bar{\alpha}}^0(0, \cdot)$ is only in H^s because it depends on v (see (4.3) and (4.4)). From (4.18)–(4.21), we conclude that $\mathcal{I}_0|_\Sigma \in H^{s-1}$. This is enough to apply the theory of quasi-linear symmetric hyperbolic systems as in [22, 34] (recall that $s > \frac{3}{2} + 2$), whose hypotheses are satisfied due to the above positive definiteness of M^t. Shrinking U if necessary, we obtain a unique solution z_U in $C^0([0, T_E'], H^{s-1}(U)) \cap C^1([0, T_E'], H^{s-2}(U))$ for some $0 < T_E' \leq T_E$.

In order to obtain the desired regularity, we shall use a bootstrap argument. Consider the system for the frame coefficients formed only by Eqs. (3.29a), where the Γ's now enter as inhomogeneous or lower-order terms given by z_U, thus they are in $C^0([0, T_E'], H^{s-1}(U)) \cap C^1([0, T_E'], H^{s-2}(U))$. This is just a first-order symmetric linear system for the frame coefficients, but there is a mismatch between the initial data, which is in H^s by (4.18), and the lower order/inhomogeneous terms, which are in H^{s-1}. The results of Fischer–Marsden [22] deal precisely with this situation, and we obtain therefore a unique $C^0([0, T_E''], H^s(U)) \cap C^1([0, T_E''], H^{s-1}(U))$ solution.[u] By uniqueness, this solution agrees with that of z_U for $[0, \min\{T_E'', T_E'\}]$, and shrinking the intervals if necessary, we can assume $T_E' = T_E''$.

Next, consider the system of equations (3.29g), (3.29h) and (3.29j). As before, the initial data, given by (4.19) is in H^s, whereas the coefficients are only in H^{s-1}. Notice that although the matrix coefficient of $\frac{\partial}{\partial t}$ is the identity, depending on the equation of state, this system is semi-linear due to the presence of the pressure in (3.29g) (see (2.12) and (2.11)). In any case, the results of [22] still apply, and we obtain that ϱ, s and r are also in H^s.

A similar argument can be applied to the system (3.29′) comprised of Eqs. (3.29b)–(3.29d). The H^{s-1} terms $E_{\bar{\alpha}\bar{\beta}}, B_{\bar{\alpha}\bar{\beta}}, s_\alpha$, and the now H^s terms ϱ, s and r given by z_U enter in the system (3.29′) as inhomogeneous or lower-order terms, whereas the initial data for (3.29′) given by $\mathcal{I}_0$ is in H^s by (4.20). We write the system once more in terms of the derivatives $\frac{\partial}{\partial x^A}$, obtaining a semi-linear system where the coefficient matrix N^t of $\frac{\partial}{\partial t}$ involves the frame coefficients $e_{\bar{\alpha}}^A$ and the sound speed

[u]The results of [22] apply to a large class of quasi-linear equations. Allowing for less regular lower-order terms was, as the authors acknowledge, one of the goals of the paper.

ν^2, which is given in terms of s and r by (2.22). From the above arguments, we obtain that N^t is positive definite and is in H^s. Evoking the results of [22] one more time gives that $\Gamma^{\bar{\beta}}_{\bar{\alpha}\bar{\gamma}}, \Gamma^0_{0\bar{\alpha}}, \Gamma^0_{\bar{\alpha}\bar{\gamma}}$ are in fact in $C^0([0,T'_E], H^s(U)) \cap C^1([0,T'_E], H^{s-1}(U))$. Using this improved regularity of the connection coefficients again with (3.29a) and (4.18), for $\bar{A} = 1,2,3$, finally gives $e^{\bar{A}}_{\bar{\alpha}} \in C^0([0,T'_E], H^{s+1}(U)) \cap C^1([0,T'_E], H^s(U))$.

We now obtain the result on $[0,T'_E] \times \Sigma$ by a standard gluing procedure. Uniqueness guarantees that solutions constructed from different patches $U$and U' agree on the domain of dependence[v] of $U \cap U'$. The time interval $[0,T'_E]$ can be made uniform due to the uniform conditions on the initial data, and the local in time, global in space, solution z will belong to the desired H^s_{ul} spaces by the way these spaces are constructed out of the Sobolev spaces of maps defined on local patches. □

The geometric meaning behind the definiteness of the matrix M^t is easy to grasp. If u were hypersurface orthogonal, then $e_{\bar{\alpha}}$ would be tangent to Σ_t, and M^t would be a diagonal matrix with positive entries. By continuity, we would expect M^t to remain positive definite as long as u is sufficiently inside the light-cone.

We also notice that the above bootstrap argument for the regularity of some of the components of z works because of the particular form of system (3.29), which can broken in several sub-systems that are "mildly coupled" among themselves. Since several of the quantities involved have direct physical meaning, it would be interesting to see if such split into sub-systems can have an useful physical interpretation, perhaps in terms of some effective notion of weak coupling among certain quantities.

Corollary 4.1. *The solutions E and B constructed in Proposition* 4.2 *are trace-free, and ϱ satisfies* (2.11).

Proof. Tracing Eqs. (3.29e) and (3.29f), we obtain a first order symmetric hyperbolic system for the traces of E and B. Since $E^\alpha_\alpha|_\Sigma = 0 = B^\alpha_\alpha|_\Sigma$ by Proposition 4.1 and (3.25), by uniqueness these tensors remain traceless.

Locally s and r are written in terms of the coordinates $s = s(x^0,\ldots,x^3), r = r(x^0,\ldots,x^3)$. From (3.29g), (3.29h), (3.29j), (2.12), we obtain

$$\frac{d}{dt}\mathscr{P}(r,s) = \partial_t \varrho,$$

which implies $\varrho = \mathscr{P}(r,s)$ since this holds at $t = 0$. □

4.3. *Propagation of the gauge*

Letting $e^A_0 = \delta^A_0$ and $\mathcal{M} = [0,T'_E] \times \Sigma$, from Proposition 4.2, we obtain a spacetime $(\mathcal{M}, g)$ with the metric given by (3.2). Notice that g agrees with g_0 on Σ

[v]Defined in the PDE sense.

because of (4.17). $(\mathcal{M}, g)$ is turned into a fluid source by setting $u = e_0$. The components of the fields that are not given in Proposition 4.2, e.g., $E_{\alpha 0}$ etc, are defined by their corresponding expressions in fluid source gauge, e.g., (3.25), and their symmetry relations. The frame $\{e_\alpha\}_{\alpha=0}^3$ is then a fluid source gauge, with the quantities of Proposition 4.2 being exactly field components written in this gauge. All other quantities *throughout this section will be written with respect to this frame* unless stated differently. Moreover, we shall also assume the hypotheses of Theorem 2.1, so that the results of the previous section will also be used throughout. The coordinates are arranged as explained after Definition 3.2; in particular $e_0 = \frac{\partial}{\partial t}$. By construction, (3.4) is satisfied and the connection associated with g is compatible with the metric, but it is not known at this point whether it is torsion free. In particular, in all expressions below involving a covariant derivative, it is to be understood that ∇ is such a connection and not the Levi-Civita one, at least until the torsion free condition is demonstrated. From E and B, we define W and W^* by (3.17) and (3.18). W has then the usual symmetries of Weyl tensor and is trace-free by Corollary 4.1, but it is not yet known that W is the Weyl tensor of the metric g. With u and ϱ known and p given by (2.12), we define $T_{\alpha\beta}$ by (2.3). Recall (3.7) and define $d^\alpha_{\beta\gamma\delta}$ via (3.5), where $S_{\alpha\beta}$ in (3.5) is given by (3.13). Define also $\mathcal{T}^\gamma_{\alpha\beta}, F^\alpha_{\beta\gamma\delta}, F_{\alpha\beta\gamma}$ and q_α, by (3.8), (3.9), (3.10) and (3.11), respectively, where ν in q_α is given by (2.22), with s, r, and ϱ being those of Proposition 4.2, which satisfy $\varrho = \mathscr{P}(r, s)$ by Corollary 4.1. We now proceed to show that (3.14b)–(3.14e) are satisfied. In order to do so, we shall derive a symmetric hyperbolic system of equations for these quantities and show that they vanish on the initial slice.

Remark 4.1. At the risk of being repetitive, we stress again that when referring to equations such as (3.17) and (3.13), it should be understood that they are being used to formally *define* $W, S_{\alpha\beta}$ etc, from the quantities obtained from Proposition 4.1. Notice also that because we do not know that ∇ is the Levi-Civita connection, the torsion tensor will have to appear in several manipulations below.

Lemma 4.1. *With the above definitions,* $d^\alpha_{\beta\gamma\delta}$ *enjoys all the symmetries of the Weyl tensor,* $\mathcal{T}^\gamma_{\alpha\beta} = -\mathcal{T}^\gamma_{\beta\alpha}$, *and* (2.5), (2.15) *and* (2.8) *are satisfied. Furthermore,* $q_0 = 0$, *and*

$$\mathcal{T}^\beta_{0\bar{\alpha}} = 0, \tag{4.22}$$

$$d^{\bar{\alpha}}_{\bar{\beta}0\bar{\gamma}} = 0, \tag{4.23}$$

$$\nabla^\mu T_{\mu\alpha} = q_\alpha. \tag{4.24}$$

Proof. The symmetries of $d^\alpha_{\beta\gamma\delta}$ and anti-symmetry of the torsion tensor are direct consequences of their definitions and the fact that $W^\alpha_{\beta\gamma\delta}$ has these symmetries. In fluid source gauge, (4.22) and (4.23) are equivalent to (3.29a) and (3.29b), respectively, while (2.5), (2.15) and (2.8) are the same as (3.29g), (3.29h) and (3.29j), respectively.

Since the pressure is a function of r and s by (2.11) and (2.12), differentiating p with respect to t and using (2.22), (3.29h) and (3.29j) yields

$$\partial_t p + (p+\varrho)\nu^2\Gamma^{\mu}_{\mu 0} = 0. \tag{4.25}$$

(4.25) is equivalent to $q_0 = 0$ in our gauge. Computing $\nabla^\mu T_{\mu\alpha}$ from (2.3) and using (3.29g) (or equivalently (2.5)),

$$\nabla^\mu T_{\mu\alpha} = (p+\varrho)u^\mu\nabla_\mu u_\alpha + u_\alpha u^\mu \nabla_\mu p - \nabla_\alpha p, \tag{4.26}$$

which in light of (4.25) and our gauge conditions, produces (4.24). □

The next lemma and the proposition that follows will be the main ingredients in proving the propagation of the gauge. Although both proofs are heavily computational, they follow the same lines of [23, 29, 30].

Lemma 4.2. *The following relations hold*:

$$\begin{aligned}
&\partial_t F_{0\bar\alpha 0} - \frac{1}{4}\varepsilon^{\bar\mu\bar\nu}_{\bar\alpha}\varepsilon^{\bar\lambda\bar\sigma}_{\bar\nu}\overline{\nabla}_{\bar\mu}F_{0\bar\lambda\bar\sigma} + d^{\mu\nu\sigma}_{[\nu}W_{\sigma]\mu\bar\alpha 0} - W_{\mu\nu\sigma[0}d^{\sigma\mu\nu}_{\bar\alpha]} \\
&\quad - \frac{1}{2}\mathcal{T}^{\sigma}_{\mu\nu}\nabla_\sigma W^{\mu\nu}_{\bar\alpha 0} + d^{\mu\nu}_{[0\bar\alpha]}S_{\mu\nu} + d^{\mu\nu}_{\nu[\bar\alpha}S_{0]\mu} + \Gamma^{\mu}_{\mu 0}F_{0\bar\alpha 0} \\
&\quad - \frac{1}{4}\Gamma^{\bar\lambda}_{\bar\alpha 0}F_{0\bar\lambda 0}\left(\nabla_0\varepsilon^{\bar\nu}_{\bar\alpha 0} - \frac{1}{2}\nabla^\mu\varepsilon^{\bar\nu}_{\mu\bar\alpha}\right)\varepsilon^{\bar\mu\bar\sigma}_{\bar\nu}F_{0\bar\mu\bar\sigma} = 0,
\end{aligned} \tag{4.27a}$$

$$\begin{aligned}
&\frac{1}{2}\varepsilon^{\bar\mu\bar\nu}_{\bar\gamma}\partial_t F_{0\bar\mu\bar\nu} + \frac{1}{2}\varepsilon^{\bar\mu\bar\nu}_{\bar\gamma}\overline{\nabla}_{\bar\mu}F_{0\bar\nu 0} + \frac{1}{2}\varepsilon^{\bar\alpha\bar\beta}_{\bar\gamma}(\Gamma^{0}_{0\bar\alpha}F_{0\bar\beta 0} - \Gamma^{0}_{0\bar\beta}F_{0\bar\alpha 0}) \\
&\quad + \frac{1}{2}\varepsilon^{\bar\alpha\bar\beta}_{\bar\gamma}\nabla^\mu\pi_{\mu[\bar\alpha}F_{|0|\bar\beta]0} - \frac{1}{4}\varepsilon^{\bar\alpha\bar\beta}_{\bar\gamma}(\varepsilon^{\bar\nu}_{\bar\alpha\bar\beta}\Gamma^{\mu}_{\mu 0} + \nabla_0\varepsilon^{\bar\nu}_{\bar\alpha\bar\beta} \\
&\quad + \varepsilon^{\bar\nu\bar\mu}_{[\bar\alpha}\Gamma^{0}_{|\bar\mu|\bar\beta]})\varepsilon^{\bar\sigma\bar\lambda}_{\bar\nu}F_{0\bar\sigma\bar\lambda} - \frac{1}{4}\mathscr{K}\varepsilon^{\bar\alpha\bar\beta}_{\bar\gamma}(\nabla^\mu\pi_{\mu\bar\alpha}q_{\bar\beta} - \nabla^\mu\pi_{\mu\bar\beta}q_{\bar\alpha}) \\
&\quad - \frac{1}{4}\mathscr{K}(p+\varrho)\nu^2\varepsilon^{\bar\alpha\bar\beta}_{\bar\gamma}d^{0}_{\bar\alpha 0\bar\beta} = 0,
\end{aligned} \tag{4.27b}$$

$$\begin{aligned}
&\partial_t\mathcal{T}^{\mu}_{\bar\beta\bar\gamma} - \sum_{(0\bar\beta\bar\gamma)}(d^{\mu}_{0\bar\beta\bar\gamma} + \mathcal{T}^{\lambda}_{0\bar\beta}\mathcal{T}^{\mu}_{\bar\gamma\lambda}) + \Gamma^{\mu}_{0\lambda}\mathcal{T}^{\lambda}_{\bar\beta\bar\gamma} \\
&\quad + \Gamma^{\mu}_{\bar\beta\lambda}\mathcal{T}^{\lambda}_{\bar\gamma 0} + \Gamma^{\mu}_{\bar\gamma\lambda}\mathcal{T}^{\lambda}_{0\bar\beta} \\
&\quad - \sum_{\langle\bar\beta\bar\gamma\rangle}\Gamma^{\lambda}_{0\bar\beta}\mathcal{T}^{\mu}_{\lambda\bar\gamma} - \sum_{\langle\bar\gamma 0\rangle}\Gamma^{\lambda}_{\bar\beta\bar\gamma}\mathcal{T}^{\mu}_{\lambda 0} - \sum_{\langle 0\bar\beta\rangle}\Gamma^{\lambda}_{\bar\gamma 0}\mathcal{T}^{\mu}_{\lambda\bar\beta} = 0,
\end{aligned} \tag{4.27c}$$

$$\partial_t d^{\bar\mu}_{\bar\nu\bar\beta\bar\gamma} + \sum_{(0\bar\beta\bar\gamma)}R^{\bar\mu}_{\bar\nu\sigma 0}\mathcal{T}^{\sigma}_{\bar\beta\bar\gamma} + \frac{1}{2}\varepsilon^{\bar\lambda}_{0\bar\beta\bar\gamma}(F_{\bar\lambda\tau\xi} + g_{\bar\lambda\tau}q_\xi)\varepsilon^{\bar\mu\tau\xi}_{\bar\nu} + \Gamma^{\bar\mu}_{0\sigma}d^{\sigma}_{\bar\nu\bar\beta\bar\gamma}$$

$$+ \Gamma^{\bar\mu}_{\bar\beta\sigma} d^{\sigma}_{\bar\nu\bar\gamma 0} + \Gamma^{\bar\mu}_{\bar\gamma\sigma} d^{\sigma}_{\bar\nu 0\bar\beta} - \sum_{\langle\bar\nu\bar\beta\bar\gamma\rangle} \Gamma^{\sigma}_{0\bar\nu} d^{\bar\mu}_{\sigma\bar\beta\bar\gamma}$$

$$- \sum_{\langle\bar\nu\bar\gamma 0\rangle} \Gamma^{\sigma}_{\bar\beta\bar\nu} d^{\bar\mu}_{\sigma\bar\gamma 0} - \sum_{\langle\bar\nu 0\bar\beta\rangle} \Gamma^{\sigma}_{\bar\gamma\bar\nu} d^{\bar\mu}_{\sigma 0\bar\beta} = 0, \tag{4.27d}$$

$$\partial_t d^{\bar\nu}_{0\bar\beta\bar\nu} - \frac{1}{2}\varepsilon^{\bar\lambda\bar\tau}_{\bar\beta}\varepsilon^{\bar\mu\bar\sigma}_{\tau} e_{\bar\lambda}(d^0_{\bar\mu 0\bar\sigma}) + \sum_{(0\bar\beta\bar\gamma)} \delta^{\bar\gamma}_{\bar\nu} R^{\bar\nu}_{0\sigma 0}\mathcal{T}^{\sigma}_{\bar\beta\bar\gamma} + \frac{1}{2}\varepsilon^{\bar\lambda}_{0\bar\beta\bar\nu}(F_{\bar\lambda\tau\xi}$$

$$+ g_{\bar\lambda\tau}q_\xi)\varepsilon^{\bar\nu\tau\xi}_0 + \delta^{\bar\gamma}_{\bar\nu}\Gamma^{\sigma}_{00} d^{\bar\nu}_{\bar\sigma\bar\beta\bar\gamma} + \delta^{\bar\gamma}_{\bar\nu}\Gamma^{\sigma}_{\bar\beta 0} d^{\bar\nu}_{\sigma\bar\gamma 0} + \delta^{\bar\gamma}_{\bar\nu}\Gamma^{\sigma}_{\bar\gamma 0} d^{\bar\nu}_{\sigma 0\bar\beta}$$

$$- \delta^{\bar\gamma}_{\bar\nu} \sum_{\langle\bar\nu\bar\beta\bar\gamma\rangle} \Gamma^{\bar\nu}_{0\sigma} d^{\sigma}_{0\bar\beta\bar\gamma} + \delta^{\bar\gamma}_{\bar\nu} \sum_{\langle\bar\nu\bar\gamma 0\rangle} \Gamma^{\bar\nu}_{\bar\beta\sigma} d^{\sigma}_{0\bar\gamma 0}$$

$$+ \delta^{\bar\gamma}_{\bar\nu} \sum_{\langle\bar\nu 0\bar\beta\rangle} \Gamma^{\bar\nu}_{\bar\gamma\sigma} d^{\sigma}_{00\bar\beta} = 0, \tag{4.27e}$$

$$\frac{1}{2}\varepsilon^{\bar\alpha\bar\beta}_{\bar\nu}\partial_t d^0_{\bar\alpha 0\bar\beta} + \frac{1}{2}\nu^2\varepsilon^{\bar\alpha\bar\beta}_{\bar\nu} e_{\bar\alpha}(d^{\bar\lambda}_{0\bar\beta\bar\lambda}) + \frac{1}{2}\nu^2\pi^{\bar\alpha\bar\lambda}\varepsilon^{\bar\sigma\bar\tau}_{(\bar\lambda} e_{1\bar\alpha}(d^0_{\bar\nu)\bar\sigma\bar\tau})$$

$$- \frac{1}{6}\nu^2\varepsilon^{\bar\beta\bar\alpha\bar\gamma}\left[\sum_{(\bar\alpha\bar\beta\bar\gamma)} R^0_{\bar\nu\sigma\bar\alpha}\mathcal{T}^{\sigma}_{\bar\beta\bar\gamma} + \frac{1}{2}\varepsilon^{\bar\lambda}_{\bar\alpha\bar\beta\bar\gamma}(F_{\bar\lambda\tau\xi} + g_{\bar\lambda\tau}q_\xi)\varepsilon^{0\tau\xi}_{\bar\nu}\right]$$

$$- \frac{1}{6}\nu^2\varepsilon^{\bar\beta\bar\alpha\bar\gamma}\Bigg[\Gamma^0_{\bar\alpha\sigma} d^{\sigma}_{\bar\nu\bar\beta\bar\gamma} + \Gamma^0_{\bar\beta\sigma} d^{\sigma}_{\bar\nu\bar\gamma\bar\alpha} + \Gamma^0_{\bar\gamma\sigma} d^{\sigma}_{\bar\nu\bar\alpha\bar\beta}$$

$$- \sum_{\langle\bar\alpha\bar\beta\bar\gamma\rangle} \Gamma^{\sigma}_{\bar\alpha\bar\nu} d^0_{\sigma\bar\beta\bar\gamma} - \sum_{\langle\bar\nu\bar\gamma\bar\alpha\rangle} \Gamma^{\sigma}_{\bar\beta\bar\nu} d^0_{\sigma\bar\gamma\bar\alpha} - \sum_{\langle\bar\nu\bar\alpha\bar\beta\rangle} \Gamma^{\sigma}_{\bar\gamma\bar\nu} d^0_{\sigma\bar\alpha\bar\beta}\Bigg]$$

$$+ \frac{1}{2}\frac{1}{p+\varrho}\varepsilon^{\bar\alpha\bar\beta}_{\bar\nu}\Bigg[\frac{1}{2}\Gamma^{\lambda}_{\sigma 0}\mathcal{T}^{\sigma}_{\bar\alpha\bar\beta} q_\lambda - \frac{1}{2}\pi^{\bar\mu}_{\bar\alpha}\pi^{\bar\nu}_{\bar\beta}(d^{\lambda}_{0\bar\mu\bar\nu} + d^{\lambda}_{\bar\mu\bar\nu 0} + d^{\lambda}_{\bar\nu 0\bar\mu})q_\lambda$$

$$+ 2\nu^2(p+\varrho)\Gamma^0_{0\bar\alpha} d^{\bar\mu}_{0\bar\beta\bar\mu} - 2\nu^2(p+\varrho)\Gamma^0_{0\bar\beta} d^{\bar\mu}_{0\bar\alpha\bar\mu} - (\partial_t p + \partial_t\varrho) d^0_{\bar\alpha 0\bar\beta}$$

$$+ \nu^2(p+\varrho)(\Gamma^{\bar\lambda}_{\bar\beta\bar\alpha} - \Gamma^{\bar\lambda}_{\bar\alpha\bar\beta}) d^{\bar\mu}_{0\bar\lambda\bar\mu} + (p+\varrho)\Gamma^{\bar\lambda}_{\bar\alpha 0} d^0_{\bar\lambda 0\bar\beta}$$

$$- (p+\varrho)\Gamma^{\bar\lambda}_{\bar\beta 0} d^0_{\bar\lambda 0\bar\alpha}$$

$$+ e_{\bar\alpha}(\nu^2(p+\varrho)) d^{\bar\mu}_{0\bar\beta\bar\mu} - e_{\bar\beta}(\nu^2(p+\varrho)) d^{\bar\mu}_{0\bar\alpha\bar\mu}\Bigg] = 0, \tag{4.27f}$$

$$\varepsilon^{\bar\lambda\bar\mu}_{(\bar\alpha}\partial_t d^0_{\bar\nu)\bar\lambda\bar\mu} - e_{(\bar\alpha}(\varepsilon^{\bar\lambda\bar\mu}_{\bar\nu)} d^0_{\bar\lambda 0\bar\mu}) + \frac{1}{2}\pi_{\bar\alpha\bar\nu}\varepsilon^{\bar\beta\bar\mu\bar\lambda} d^0_{\bar\mu 0\bar\lambda} e_{\bar\beta}\left(\frac{1}{p+\varrho}\right)$$

$$- \frac{1}{2}\frac{1}{p+\varrho}\pi_{\bar\alpha\bar\nu}\varepsilon^{\bar\lambda\bar\mu\bar\sigma} R^{\tau}_{\bar\lambda\bar\mu\bar\sigma} q_\tau - \frac{1}{2}\pi_{\bar\alpha\bar\nu}\varepsilon^{\bar\beta\bar\mu\bar\lambda}(\nu^2\Gamma^0_{\bar\beta\bar\mu} d^{\bar\xi}_{0\bar\lambda\bar\xi} - \Gamma^{\bar\sigma}_{\bar\beta\bar\mu} d^0_{\bar\sigma 0\bar\lambda}$$

$$+ \nu^2\Gamma^0_{\bar\beta\bar\lambda} d^{\bar\xi}_{0\bar\mu\bar\xi} - \Gamma^{\bar\sigma}_{\bar\beta\bar\lambda} d^0_{\bar\mu 0\bar\sigma}) + \frac{1}{2}\varepsilon^{\bar\beta\bar\gamma}_{(\bar\alpha} \sum_{(0\bar\beta\bar\gamma)} R^0_{\bar\nu)\sigma 0}\mathcal{T}^{\sigma}_{\bar\beta\bar\gamma}$$

$$+\frac{1}{4}\varepsilon^{\bar{\beta}\bar{\gamma}}_{(\bar{\alpha}}\varepsilon^{0\tau\xi}_{\bar{\nu})}\varepsilon^{\lambda}_{0\bar{\beta}\bar{\gamma}}(F_{\lambda\tau\xi}+g_{\lambda\tau}q_{\xi})+\frac{1}{2}\varepsilon^{\bar{\beta}\bar{\gamma}}_{(\bar{\alpha}}(d^{\sigma}_{\bar{\nu})\bar{\beta}\bar{\gamma}}\Gamma^{0}_{0\sigma}$$

$$+d^{\sigma}_{\bar{\nu})\bar{\gamma}0}\Gamma^{0}_{\bar{\beta}\sigma}+d^{\sigma}_{\bar{\nu})0\bar{\beta}}\Gamma^{0}_{\bar{\gamma}\sigma})+\frac{1}{2}\varepsilon^{\bar{\beta}\bar{\gamma}}_{(\bar{\alpha}}\left(\sum_{\langle\bar{\nu}\bar{\beta}\bar{\gamma}\rangle}\Gamma^{\sigma}_{|0|\bar{\nu})}d^{0}_{\sigma\bar{\beta}\bar{\gamma}}\right.$$

$$\left.+\sum_{\langle\bar{\nu}\bar{\gamma}0\rangle}\Gamma^{\sigma}_{|\bar{\beta}|\bar{\nu})}d^{0}_{\sigma\bar{\gamma}0}+\sum_{\langle\bar{\nu}0\bar{\beta}\rangle}\Gamma^{\sigma}_{|\bar{\gamma}|\bar{\nu})}d^{0}_{\sigma0\bar{\beta}}\right)=0, \tag{4.27g}$$

$$\partial_t q_{\bar{\alpha}}+(\Gamma^{\bar{\lambda}}_{\bar{\alpha}0}-\Gamma^{\bar{\lambda}}_{0\bar{\alpha}})q_{\bar{\lambda}}-2\nu^2(p+\varrho)d^{\bar{\gamma}}_{0\bar{\alpha}\bar{\gamma}}=0, \tag{4.27h}$$

where $(\cdot)_{(\alpha|\mu|\beta)}$(*respectively* $(\cdot)_{[\alpha|\mu|\beta]}$) *means symmetrization* (*respectively antisymmetrization*) *of the indices* α *and* β *only,* $\sum_{(\alpha\beta\gamma)}$ *indicates sum over cyclic permutations of* $\alpha\beta\gamma$, *and*

$$\textstyle\sum_{\langle\alpha\beta\rangle}\Gamma^{\lambda}_{\tau\alpha}A_{\lambda\beta}:=\Gamma^{\lambda}_{\tau\alpha}A_{\lambda\beta}+\Gamma^{\lambda}_{\tau\beta}A_{\alpha\lambda},$$

$$\textstyle\sum_{\langle\alpha\beta\gamma\rangle}\Gamma^{\lambda}_{\tau\alpha}A_{\lambda\beta\gamma}:=\Gamma^{\lambda}_{\tau\alpha}A_{\lambda\beta\gamma}+\Gamma^{\lambda}_{\tau\beta}A_{\alpha\lambda\gamma}+\Gamma^{\lambda}_{\tau\gamma}A_{\alpha\beta\lambda}.$$

Proof. We start computing $\nabla^{\mu}F_{\mu\alpha\beta}$. Commuting the covariant derivatives and using the symmetries of $W^{\mu}_{\alpha\beta\gamma}$ and $S_{\alpha\beta}$ we find

$$\begin{aligned}2\nabla^{\mu}F_{\mu\alpha\beta}&=-R^{\mu\nu\sigma}_{\nu}W_{\sigma\mu\alpha\beta}+R^{\mu\nu\sigma}_{\sigma}W_{\nu\mu\alpha\beta}+R^{\sigma\mu\nu}_{\alpha}W_{\mu\nu\sigma\beta}\\&\quad-R^{\sigma\mu\nu}_{\beta}W_{\mu\nu\sigma\alpha}+\mathcal{T}^{\sigma}_{\mu\nu}\nabla_{\sigma}W^{\mu\nu}_{\alpha\beta}+\nabla_{\alpha}\nabla^{\mu}S_{\beta\mu}-\nabla_{\beta}\nabla^{\mu}S_{\alpha\mu}\\&\quad-R^{\mu\nu}_{\beta\alpha}S_{\mu\nu}+R^{\mu\nu}_{\alpha\beta}S_{\mu\nu}-R^{\mu\nu}_{\nu\alpha}S_{\beta\mu}+R^{\mu\nu}_{\nu\beta}S_{\alpha\mu}\\&\quad+\nabla_{\mu}S^{\nu}_{\beta}\mathcal{T}^{\mu}_{\alpha\nu}-\nabla_{\mu}S^{\nu}_{\alpha}\mathcal{T}^{\mu}_{\beta\nu}.\end{aligned}$$

Using (3.5), (3.13) and (4.24) and the various symmetries of the tensors involved, the above becomes (recall that T is the trace of $T_{\alpha\beta}$)

$$\begin{aligned}\nabla^{\mu}F_{\mu\alpha\beta}&=-d^{\mu\nu\sigma}_{[\nu}W_{\sigma]\mu\alpha\beta}+W_{\mu\nu\sigma[\beta}d^{\sigma\mu\nu}_{\alpha]}+\frac{1}{2}\mathcal{T}^{\sigma}_{\mu\nu}\nabla_{\sigma}W^{\mu\nu}_{\alpha\beta}\\&\quad-d^{\mu\nu}_{[\beta\alpha]}S_{\mu\nu}-W^{\mu\nu}_{[\beta\alpha]}S_{\mu\nu}+d^{\mu\nu}_{\nu[\alpha}S_{\beta]\nu}\\&\quad+\frac{1}{2}\mathscr{K}[(\nabla_{\mu}p+\nabla_{\mu}\varrho)(\mathcal{T}^{\mu}_{\alpha0}\delta_{0\beta}+\mathcal{T}^{\mu}_{\beta0}\delta_{0\alpha})\\&\quad+\nabla_{\mu}p(\mathcal{T}^{\mu}_{\beta\alpha}-\mathcal{T}^{\mu}_{\alpha\beta})]\\&\quad-\frac{1}{6}\mathscr{K}(\mathcal{T}^{\mu}_{\alpha\beta}-\mathcal{T}^{\mu}_{\beta\alpha})\nabla_{\mu}T+\frac{1}{6}\mathscr{K}\mathcal{T}^{\mu}_{\alpha\beta}\nabla_{\mu}T+\mathscr{K}\nabla_{[\alpha}q_{\beta]}.\end{aligned} \tag{4.28}$$

On the other hand, using (4.24) into $F_{\mu\alpha\beta}$ gives, after some contractions,

$$-\frac{1}{2}\mathscr{K}q_{\alpha}=\pi^{\sigma}_{\mu}\pi^{\tau}_{\nu}\pi^{\lambda}_{\alpha}F_{\sigma\tau\lambda}\pi^{\mu\nu}-\pi^{\lambda}_{\alpha}F_{\mu\lambda\nu}u^{\mu}u^{\nu}. \tag{4.29}$$

Then, from (3.17), (3.29e), (3.29f), (3.29g) and (4.29), we obtain, after some algebra,

$$\begin{aligned} F_{\mu\alpha\beta} &= u_\mu \pi_\alpha^\lambda F_{\sigma\lambda\nu} u^\sigma u^\nu u_\beta - u_\mu \pi_\beta^\lambda F_{\sigma\lambda\nu} u^\sigma u^\nu u_\alpha + \frac{1}{2}\pi_{\mu\alpha}\pi_\beta^\lambda F_{\sigma\lambda\nu} u^\sigma u^\nu \\ &\quad - \frac{1}{2}\pi_{\mu\beta}\pi_\alpha^\lambda F_{\sigma\lambda\nu} u^\sigma u^\nu - \frac{1}{2} u_\mu \varepsilon_{\alpha\beta}^{\nu} \varepsilon_\nu^{\sigma\tau} \pi_\sigma^\lambda \pi_\tau^\xi F_{\gamma\lambda\xi} u^\gamma \\ &\quad + \frac{1}{2}\varepsilon_{\mu[\alpha}^{\nu} u_{\beta]} \varepsilon_\nu^{\sigma\tau} \pi_\sigma^\lambda \pi_\tau^\xi F_{\gamma\lambda\xi} u^\gamma - \frac{1}{2}\mathscr{K}\pi_{\mu[\alpha} q_{\beta]}. \end{aligned} \tag{4.30}$$

Computing $\nabla^\mu F_{\mu\alpha\beta}$ from (4.30), using the resulting expression into (4.28), recalling our gauge conditions, and evoking Lemma 4.1 leads to (4.27a) and (4.27b) after suitably choosing the indices to correspond to the ones of those expressions.

When the torsion of ∇ does not necessarily vanish, the first Bianchi identity takes the form

$$\sum_{(\alpha\beta\gamma)} R^\mu_{\alpha\beta\gamma} = \sum_{(\alpha\beta\gamma)} (\nabla_\alpha \mathcal{T}^\mu_{\beta\gamma} - \mathcal{T}^\lambda_{\alpha\beta}\mathcal{T}^\mu_{\gamma\lambda}). \tag{4.31}$$

After evoking (3.17), using symmetries, and setting $\alpha = 0$, $\beta = \bar\beta$ and $\gamma = \bar\gamma$, this expression simplifies to

$$\nabla_0 \mathcal{T}^\mu_{\bar\beta\bar\gamma} = -\nabla_{\bar\beta}\mathcal{T}^\mu_{\bar\gamma 0} - \nabla_{\bar\gamma}\mathcal{T}^\mu_{0\bar\beta} + \sum_{(0\bar\beta\bar\gamma)} (d^\mu_{0\bar\beta\bar\gamma} + \mathcal{T}^\lambda_{0\bar\beta}\mathcal{T}^\mu_{\bar\gamma\lambda}).$$

In view of (4.22), this gives (4.27c).

Taking torsion into account again, the second Bianchi identity reads

$$\sum_{(\alpha\beta\gamma)} \nabla_\alpha R^\mu_{\nu\beta\gamma} + \sum_{(\alpha\beta\gamma)} R^\mu_{\nu\lambda\alpha}\mathcal{T}^\lambda_{\beta\gamma} = 0. \tag{4.32}$$

From the symmetries of $W^\alpha_{\beta\gamma\delta}$, $d^\alpha_{\beta\gamma\delta}$ and (4.31), we have the identity

$$\sum_{(\alpha\beta\gamma)} \nabla_\alpha W^\mu_{\nu\beta\gamma} + \frac{1}{2}\varepsilon_\nu^{\lambda\sigma\mu}\,\varepsilon^\xi_{\alpha\beta\gamma}\nabla_\tau W^\tau_{\xi\lambda\sigma} = 0. \tag{4.33}$$

From (4.32), (3.5), (3.17), (3.18), (4.33), (3.13), and (4.24), we get, after some simplifications,

$$\sum_{(\alpha\beta\gamma)} \nabla_\alpha d^\mu_{\nu\beta\gamma} + \sum_{(\alpha\beta\gamma)} R^\mu_{\nu\lambda\alpha}\mathcal{T}^\lambda_{\beta\gamma} + \frac{1}{2}\varepsilon^\lambda_{\alpha\beta\gamma}(F_{\lambda\tau\xi} + g_{\lambda\tau}q_\xi)\varepsilon_\nu^{\mu\tau\xi} = 0. \tag{4.34}$$

Setting $\alpha = 0$, $\mu = \bar\mu$, $\nu = \bar\nu$, $\beta = \bar\beta$ and $\gamma = \bar\gamma$ in (4.34) and using (4.23) produces (4.27d).

From the definition of q_α, compute $\nabla_{[\bar\alpha} q_{\bar\beta]}$, use (3.29d) and $\nu^2 > 0$ to find

$$(p + \varrho) d^0_{\bar\alpha 0\bar\beta} = -\nabla_{[\bar\alpha} q_{\bar\beta]}, \tag{4.35}$$

which implies

$$d^0_{\bar\alpha 0\bar\beta} + d^0_{\bar\beta 0\bar\alpha} = 0. \tag{4.36}$$

Then (4.36), (3.15) and the symmetries of $d^\alpha_{\beta\gamma\delta}$ imply the identity

$$g^{\bar\gamma\bar\nu} e_{\bar\gamma}(d^0_{\bar\nu 0\bar\beta}) - g^{\bar\gamma\bar\nu} e_{\bar\beta}(d^0_{\bar\nu 0\bar\gamma}) = -\frac{1}{2}\varepsilon^{\bar\lambda\bar\tau}_{\bar\beta} e_{\bar\lambda}(\varepsilon^{\bar\mu\bar\xi}_{\bar\tau} d^0_{\bar\mu 0\bar\xi}). \tag{4.37}$$

From (4.34) with $\alpha = 0, \mu = 0, \nu = \bar\nu, \beta = \bar\beta$ and $\gamma = \bar\gamma$,

$$\begin{aligned} &e_0(d^0_{\bar\nu\bar\beta\bar\gamma}) + e_{\bar\beta}(d^0_{\bar\nu\bar\gamma 0}) + e_{\bar\gamma}(d^0_{\bar\nu 0\bar\beta}) + \sum_{(0\bar\beta\bar\gamma)} R^0_{\bar\nu\sigma 0}\mathcal{T}^\sigma_{\bar\beta\bar\gamma} + \Gamma^0_{0\sigma} d^\sigma_{\bar\nu\bar\beta\bar\gamma} \\ &\quad + \Gamma^0_{\bar\beta\sigma} d^\sigma_{\bar\nu\bar\gamma 0} + \Gamma^0_{\bar\gamma\sigma} d^\sigma_{\bar\nu 0\bar\beta} - \sum_{\langle\bar\nu\bar\beta\bar\gamma\rangle} \Gamma^\sigma_{0\bar\nu} d^0_{\sigma\bar\beta\bar\gamma} - \sum_{\langle\bar\nu\bar\gamma 0\rangle} \Gamma^\sigma_{\bar\beta\bar\nu} d^0_{\sigma\bar\gamma 0} \\ &\quad - \sum_{\langle\bar\nu 0\bar\beta\rangle} \Gamma^\sigma_{\bar\gamma\bar\nu} d^0_{\sigma 0\bar\beta} + \frac{1}{2}\varepsilon^{\bar\lambda}_{0\bar\beta\bar\gamma}(F_{\bar\lambda\tau\xi} + g_{\bar\lambda\tau} q_\xi)\varepsilon^{0\tau\xi}_{\bar\nu} = 0. \end{aligned} \tag{4.38}$$

Using (3.21)–(3.24), observing that $R^0_{\bar\nu\sigma\lambda} = R^{\bar\nu}_{0\sigma\lambda}$, $d^0_{\bar\nu\sigma\lambda} = d^{\bar\nu}_{0\sigma\lambda}$, using (4.37) into (4.38), and contracting in $\bar\nu$ and $\bar\gamma$ produces (4.27e).

From (3.29c) and the definition of q_α, we find

$$(p + \varrho)\nu^2 d^{\bar\nu}_{0\bar\alpha\bar\nu} = \nabla_{[0} q_{\bar\alpha]}. \tag{4.39}$$

From (4.35) and (4.39), we then obtain

$$e_0(\nabla_{[\bar\alpha} q_{\bar\beta]}) = -d^0_{\bar\alpha 0\bar\beta} e_0(p + \varrho) - (p + \varrho) e_0(d^0_{\bar\alpha 0\bar\beta}), \tag{4.40}$$

$$\nabla_{\bar\alpha}\nabla_{[0} q_{\bar\beta]} = e_{\bar\alpha}(\nabla_{[0} q_{\bar\beta]}) - \Gamma^\lambda_{\bar\alpha\bar\beta}\nabla_{[0} q_{\lambda]} - \Gamma^\lambda_{\bar\alpha 0}\nabla_{[\lambda} q_{\bar\beta]}, \tag{4.41}$$

$$\nabla_{\bar\beta}\nabla_{[0} q_{\bar\alpha]} = e_{\bar\beta}(\nabla_{[0} q_{\bar\alpha]}) - \Gamma^\lambda_{\bar\beta\bar\alpha}\nabla_{[0} q_{\lambda]} - \Gamma^\lambda_{\bar\beta 0}\nabla_{[\lambda} q_{\bar\alpha]}. \tag{4.42}$$

Recall the following identity for the Lie derivative $\mathcal{L}$

$$\mathcal{L}_{e_0}\nabla_{[\bar\alpha} q_{\bar\beta]} = \nabla_0\nabla_{[\bar\alpha} q_{\bar\beta]} + \nabla_{[\bar\alpha} q_{\mu]}\Gamma^\mu_{\bar\beta 0} + \nabla_{[\mu} q_{\bar\beta]}\Gamma^\mu_{\bar\alpha 0}. \tag{4.43}$$

Now compute the left-hand side of (4.43) directly from (3.11), use (4.40)–(4.42), contract the resulting expression with $\varepsilon^{\bar\alpha\bar\beta}_{\bar\nu}$, and evoke (4.34) once again to find (4.27f).

Next, set $\alpha = 0, \mu = 0, \nu = \bar\nu, \beta = \bar\beta$ and $\gamma = \bar\gamma$ in (4.34), contract with $\varepsilon^{\bar\beta\bar\gamma}_{\bar\alpha}$, symmetryze on $\bar\alpha$ and $\bar\nu$, and use (4.35) and (4.39) to obtain (4.27g).

Finally, (4.27h) follows from (4.39) and Lemma 4.1. □

Next, we show that the tensors $d^\alpha_{\beta\gamma\delta}$, $F_{\alpha\beta\gamma}$ and $\mathcal{T}^\gamma_{\alpha\beta}$ vanish on $\mathcal{M}$.

Proposition 4.3. *With the above definitions,*

$$d^\alpha_{\beta\gamma\delta} = 0, F_{\alpha\beta\gamma} = 0, \mathcal{T}^\gamma_{\alpha\beta} = 0, \text{ and } q_\alpha = 0$$

on $\mathcal{M}$.

Proof. In light of our gauge choice, Lemma 4.1, and the symmetries involved, several components of the above tensors vanish identically on $\mathcal{M}$. Taking into account the symmetries of the remaining components, it is seen that to show the proposition it suffices to prove that the components

$$\mathcal{T}^{\gamma}_{\bar{\alpha}\bar{\beta}}, F_{0\bar{\alpha}0}, F_{0\bar{\alpha}\bar{\beta}}, F_{\bar{\alpha}\bar{\beta}\bar{\gamma}}, d^{0}_{\bar{\alpha}0\bar{\beta}}, d^{0}_{\bar{\alpha}\bar{\beta}\bar{\gamma}}, d^{\bar{\mu}}_{\bar{\alpha}\bar{\beta}\bar{\gamma}} \text{ and } q_{\bar{\alpha}} \tag{4.44}$$

vanish on $\mathcal{M}$.

Recalling that $F_{\alpha\beta\gamma} = -F_{\alpha\gamma\beta}$, Eqs. (4.27) can be viewed as a system for the quantities

$$\mathcal{T}^{\gamma}_{\bar{\alpha}\bar{\beta}}, F_{0\bar{\alpha}0}, F_{0\bar{\alpha}\bar{\beta}}, d^{\bar{\mu}}_{\bar{\alpha}\bar{\beta}\bar{\gamma}}, d^{\bar{\nu}}_{0\bar{\alpha}\bar{\nu}}, \varepsilon^{\bar{\alpha}\bar{\beta}}_{\bar{\nu}} d^{0}_{\bar{\alpha}0\bar{\beta}}, \varepsilon^{\bar{\beta}\bar{\gamma}}_{(\bar{\nu}} d^{0}_{\bar{\alpha})\bar{\beta}\bar{\gamma}} \text{ and } q_{\bar{\alpha}}. \tag{4.45}$$

Arguing as in the proof of Proposition 4.2, we obtain (after multiplying Eq. (4.27e) by ν^2) that (4.27) is a first order symmetric hyperbolic system. Uniqueness of solutions then implies that all quantities in (4.45) vanish on $\mathcal{M}$ if they vanish on Σ. To show that this is the case, it is useful to introduce the adapted coordinates $\{\widetilde{x}^A\}_{A=0}^3$ and frame $\{\widetilde{e}_\mu\}_{\mu=0}^3$ as in the proof of Proposition 4.1, and we shall employ the same notation and conventions as used there.[w]

From (4.22), it follows that

$$\widetilde{\mathcal{T}}^{\gamma}_{\alpha\beta}\widetilde{u}^{\alpha} = 0.$$

Using (4.5), this gives

$$\widetilde{\mathcal{T}}^{\gamma}_{0\bar{\beta}}\Lambda^0_0 + \widetilde{\mathcal{T}}^{\gamma}_{\bar{\alpha}\bar{\beta}}\Lambda^{\bar{\alpha}}_0 = 0.$$

But since the connection on three manifold (Σ, g_0) is torsion free, we have $\widetilde{\mathcal{T}}^{\bar{\gamma}}_{\bar{\alpha}\bar{\beta}}|_\Sigma = 0$, whereas $\widetilde{\mathcal{T}}^{0}_{\bar{\alpha}\bar{\beta}}|_\Sigma = 0$ by (4.1). Therefore, $\widetilde{\mathcal{T}}^{\gamma}_{0\bar{\beta}}|_\Sigma = 0$ since $\Lambda^0_0 \neq 0$. We conclude that $\widetilde{\mathcal{T}}^{\gamma}_{\alpha\beta}$, and thus $\mathcal{T}^{\gamma}_{\alpha\beta}$, vanishes on Σ.

From (4.30) and the way the initial data was constructed in proposition (4.1), we find, after a somewhat lengthy but not difficult calculation, that

$$F_{0\bar{\alpha}0} = 0 = F_{0\bar{\alpha}\bar{\beta}} \quad \text{on } \Sigma. \tag{4.46}$$

Using Corollary 4.1 and Lemma 3.1, we obtain, with the help of (4.24),

$$g^{\alpha\beta}F_{\alpha\beta\gamma} = -\frac{1}{2}\mathscr{K} q_\gamma.$$

In light of (4.46) and (4.30), this gives

$$q_{\bar{\gamma}} = 0 \quad \text{on } \Sigma,$$

which then implies, upon employing (4.30) one more time, that

$$F_{\bar{\alpha}\bar{\beta}\bar{\gamma}} = 0 \quad \text{on } \Sigma,$$

where (4.46) has been used.

[w]As some identities of the proof of Proposition 4.1 will be evoked as well, it should be noticed that these are still valid assuming only the hypotheses of this section.

Finally, using the Gauss equation and arguing as in Proposition 4.1, we conclude that the constraints implied by the decomposition of the Riemann tensor (i.e. (3.5) with $d^\alpha_{\beta\gamma\delta} = 0$) are satisfied on Σ. This, combined with (4.23), gives the vanishing of $d^\alpha_{\beta\gamma\delta}$ on Σ.

We conclude that the quantities (4.45) vanish on $\mathcal{M}$. This implies, evoking (4.30) once more, that $F_{\bar\alpha\bar\beta\bar\gamma} = 0$ also holds on $\mathcal{M}$. The remaining components in (4.44) also vanish due to the identities

$$-2d^0_{\bar\alpha 0\bar\beta} = \varepsilon^{\bar\mu}_{\bar\alpha\bar\beta}\varepsilon^{\bar\sigma\bar\tau}_{\bar\mu} d^0_{\bar\sigma 0\bar\tau},$$

$$-2d^0_{\bar\nu\bar\lambda\bar\mu} = \varepsilon^{\bar\alpha}_{\bar\mu\bar\lambda}\varepsilon^{\bar\beta\bar\gamma}_{(\bar\nu} d^0_{\bar\alpha)\bar\beta\bar\gamma} + \pi_{\bar\nu\bar\mu} d^{\bar\gamma}_{0\bar\lambda\bar\gamma} - \pi_{\bar\nu\bar\lambda} d^{\bar\gamma}_{0\bar\mu\bar\gamma},$$

which are verified by inspection with the help of (4.36), (3.15) and (3.16). □

Remark 4.2. Notice that, as in the familiar case of wave coordinates, propagation of the gauge requires that the constraint equations be satisfied.

We already know that the connection associated with $\Gamma^\gamma_{\alpha\beta}$ is metric. Since it is also torsion-free by Proposition 4.3, we obtain:

Corollary 4.2. *The connection defined by the coefficients $\Gamma^\gamma_{\alpha\beta}$ is the Levi-Civita connection of the metric g.*

4.4. *Solution to the original system*

Showing that the solution z of the reduced system yields a solution to the original Einstein–Euler-Entropy system is now a matter of unwrapping all our definitions.

Proposition 4.4. *Let $W^\alpha_{\beta\gamma\delta}$ and $S_{\alpha\beta}$ be as in Sec.* 4.3 *and g the metric constructed out of the solution of the reduced system given in Proposition* 4.2. *Then $W^\alpha_{\beta\gamma\delta}$ and $S_{\alpha\beta}$ are, respectively, the Weyl and the Schouten tensor of the metric g. Furthermore, Eqs.* (3.14) *are satisfied, and the quantity s_α from Proposition* 4.2 *is in fact the derivative of s.*

Proof. The connection given by $\Gamma^\gamma_{\alpha\beta}$ is the Levi-Civita connection of g by Corollary 4.2. Thus, if we denote by $\widehat{W}^\alpha_{\beta\gamma\delta}$ and $\widehat{S}_{\alpha\beta}$ the Weyl and the Schouten tensor of the metric g, we have

$$R^\alpha_{\beta\gamma\delta} = \widehat{W}^\alpha_{\beta\gamma\delta} + g^\alpha_{[\gamma}\widehat{S}_{\delta]\beta} - g_{\beta[\gamma}\widehat{S}^\alpha_{\delta]}$$

which in turn equals $W^\alpha_{\beta\gamma\delta} + g^\alpha_{[\gamma}S_{\delta]\beta} - g_{\beta[\gamma}S^\alpha_{\delta]}$ since $d^\alpha_{\beta\gamma\delta} = 0$ by Proposition 4.3. Hence, tracing the equality

$$\widehat{W}^\alpha_{\beta\gamma\delta} + g^\alpha_{[\gamma}\widehat{S}_{\delta]\beta} - g_{\beta[\gamma}\widehat{S}^\alpha_{\delta]} = W^\alpha_{\beta\gamma\delta} + g^\alpha_{[\gamma}S_{\delta]\beta} - g_{\beta[\gamma}S^\alpha_{\delta]} \tag{4.47}$$

and using that W is traceless in light of Corollary 4.1, we obtain that $S_{\alpha\beta}$ is indeed the Schouten tensor, which then implies $W^\alpha_{\beta\gamma\delta} = \widehat{W}^\alpha_{\beta\gamma\delta}$ by using (4.47) again. Then (3.13), with $S_{\alpha\beta}$ being the Schouten tensor, also holds. The remaining equations of (3.14) are satisfied by Propositions 4.2 and 4.3, and Corollary 4.1. Notice that these results show the validity of the (3.14) in fluid source gauge, but by the tensorial nature of the equations, they hold in any frame.

Put $\widehat{s}_\alpha = \nabla_\alpha s$. Then, since $u^\alpha \nabla_\alpha s = 0$ by (3.29h), we obtain $\mathcal{L}_u \widehat{s}_\alpha = 0$, where $\mathcal{L}$ is the Lie derivative, which in turn implies

$$\partial_t \widehat{s}_\alpha - (\Gamma^\mu_{0\alpha} - \Gamma^\mu_{\alpha 0})\widehat{s}_\mu = 0.$$

From (3.29i) and the construction of the initial data (Proposition 4.1), it follows that $\widehat{s}_\alpha = s_\alpha$. □

Proof of Theorem 2.1. By Proposition 4.4 and (4.24) we obtain that the equations of the Einstein–Euler-Entropy system are satisfied. Notice that $u^\alpha u_\alpha = 1$ holds by the way the solution was constructed; so $\nabla^\mu T_{\mu\alpha} = 0$ in fact implies (2.5) and (2.6). That $\mathcal{M} \approx [0, T_E] \times \Sigma$ is indeed an Einsteinian development follows from the fact that the constraint equations are satisfied, and by construction $(\mathcal{M}, g, u)$ is a perfect fluid source.

Let $\{e_\mu\}_{\mu=0}^3$ be a perfect fluid source gauge, with the coordinates $\{x^A\}_{A=0}^3$ arranged as explained below Definition 3.2. Since $e^{\bar{A}}_{\bar{\alpha}}$ is in $C^0([0, T_E], H^{s+1}_{ul}(\Sigma)) \cap C^1([0, T_E], H^s_{ul}(\Sigma))$ and $e^A_0 = \delta^A_0$, by (3.2) we conclude that the inverse of the one-parameter family of metrics g_t induced on $\Sigma_t = \{t = \text{constant}\}$ belongs to $C^0([0, T_E], H^{s+1}_{ul}(\Sigma)) \cap C^1([0, T_E], H^s_{ul}(\Sigma))$, and so does g_t itself because $s > \frac{3}{2} + 2$. Since $d^\alpha_{\beta\gamma\delta} = 0$ and $E_{\bar{\alpha}\bar{\beta}}$, $B_{\bar{\alpha}\bar{\beta}} \in C^0([0, T_E], H^{s+1}_{ul}(\Sigma)) \cap C^1([0, T_E], H^s_{ul}(\Sigma)) \cap C^2([0, T_E], H^{s-1}_{ul}(\Sigma))$, by (3.17), (3.25), (3.25), Propositions 4.3 and 4.2, we conclude that $g \in C^0([0, T_E], H^{s+1}_{ul}(\Sigma)) \cap C^1([0, T_E], H^s_{ul}(\Sigma)) \cap C^2([0, T_E], H^{s-1}_{ul}(\Sigma))$. The four-velocity u belongs to $C^0([0, T_E], H^s_{ul}(\Sigma)) \cap C^1([0, T_E], H^{s-s}_{ul}(\Sigma))$ and has the correct projection because of (4.4) and Proposition 4.2.

The functions r and s have the desired regularity and take the correct initial values by the way they have been constructed, and ϱ and p are given by (2.11) and (2.12) as a consequence of Corollary 4.1, which implies that ν^2 also has the correct form. By continuity on the time variable and the hypotheses of the theorem, we obtain that $r > 0$ and $\nu^2 > 0$ for small T_E. We cannot have $s(p) < 0$ for a point p near the initial Cauchy surface because this would contradict $u^\alpha \nabla_\alpha s = 0$ and $\varsigma_0 \geq 0$, hence $s \geq 0$. □

Remark 4.3. The less regular frame coefficients $e^0_{\bar{\alpha}}$ do not affect the regularity of the space-time metric because they contribute only to the mixed entries $g_{0\bar{A}}$, which are gauge terms therefore having no direct physical or geometrical meaning.

5. Further Remarks

The case of barotropic fluids is treated as a particular case of Theorem 2.1, at least as long as (2.23a) and (2.23c) hold. Although the condition $\nu^2 > 0$ is violated by pressure-free matter, in this situation, Eqs. (3.14) simplify considerably; a reduced system which does not require the introduction of ν^2 can be derived [23], and a system for the propagation of the gauge, which does not involve ν^2 either can also be constructed [30]. The arguments of Sec. 4 can then be reproduced, yielding a statement analogous to Theorem 2.1 for pressure-free matter. Notice also that when a fluid is isentropic, the entropy can be treated as a parameter in the equation of state, and the equations of motion take the form of those of a barotropic fluid.

It should also be noticed that condition (2.23b) has never been used. In fact, such inequality is necessary due to causality, but it plays no role on the well-posedness of the Einstein–Euler-Entropy system. There is at least one situation where it may be desirable to consider equations of state where $\nu^2 \leq 1$ is not satisfied, namely, the construction of appropriate gauge conditions for the vacuum Einstein equations. In this situation, setting $\mathscr{K} = 0$ in our system, the Euler equations decouple, and the role of the four-velocity u is to fix the gauge. As Friedrich has pointed out [23], this procedure may be particularly important in numerical treatments of Einstein equations, where ever more sophisticated gauge choices are crucial for accurate results.

Finally, we remark that although Theorem 2.1 does not provide an existence result for the fluid body discussed in the introduction, when one is interested solely in its behavior near a small compact set $\Omega \subset \Sigma$, the hypothesis that r_0 is uniformly bounded away from zero can be replaced by the condition that r_0 is positive in Ω and decays sufficiently fast on its complement. As mentioned in Sec. 2, this will not generally yield a uniform time span for the solution, but a uniform $T_E > 0$ will exist in the neighborhood of Ω. Whether this suffices for studying the dynamics of the fluid near Ω will obviously depend on the particular application one has in mind.

Acknowledgment

The author was partially supported by NSF grant 1305705.

Appendix A. The Uniformly Local Sobolev Spaces

Below we review the uniformly local Sobolev spaces originally introduced by Kato [34]. Some of their properties can also be found in [8].

Let (M, γ) be a Riemannian manifold. For any open set $U \subseteq M$, let $H^s(U, \gamma)$ be the Sobolev space of tensors of a given rank defined on U, with the derivative ∇ and the measured of integration μ used to define the Sobolev norm being those of the metric γ. Recall that $H^s_{\text{loc}}(M, \gamma)$ is defined as the space of tensor fields that belong to $H^s(U, \gamma)$ for any relatively compact $U \subseteq M$.

Definition A.1. Let $\{U_j\}$ be a locally finite covering of M by relatively compact open sets U_j. We define $H^s_{ul}(M, \gamma)$ as the space of tensor fields $f \in H^s_{\text{loc}}(M, \gamma)$ such that

$$\|f\|_{H^s_{ul}} := \sup_j \|f\|_{H^s(U_j, \gamma)} < \infty.$$

$H^s_{ul}(M,\gamma)$ is a Banach space with the above norm. $H^s_{ul}(M,\gamma)$ will have the usual embedding and multiplication properties of Sobolev spaces provided all of the $H^s(U_j,\gamma)$ have them. This will be the case when (M,γ) has injectivity radius bounded from below away from zero. In this case, it also holds that the Sobolev constants relative to the subsets U_i are uniformly bounded, and $\gamma|_{U_i}$ is uniformly equivalent to the Euclidean metric.

Appendix B. Derivation of the Reduced System

In this appendix, it is shown how Eqs. (3.29) are obtained from (3.14) after making a gauge choice. This has been done first by Friedrich in [23], which the reader is referred to for more details.

Assume that a solution to the Einstein–Euler-Entropy system in the frame formalism, Eqs. (3.14), is given.

In fluid source gauge, the vanishing of $\mathcal{T}^{\mu}_{0\bar{\alpha}}$ corresponds to Eq. (3.29a), while (3.7), (3.17) and $d^{\alpha}_{\beta\gamma\delta}=0$ give (3.29b).

It follows from our definitions that

$$\begin{aligned}\frac{1}{p+\varrho}\nabla_{[\alpha}q_{\beta]} &= u^\mu\nabla_\alpha\nabla_\mu u_\beta - u^\mu\nabla_\beta\nabla_\mu u_\alpha - \nu^2 u_\beta\nabla_\alpha\nabla_\mu u^\mu + \nu^2 u_\alpha\nabla_\beta\nabla_\mu u^\mu \\ &\quad - \nu^2\nabla_\mu u^\mu\nabla_\alpha u_\beta + \nu^2\nabla_\mu u^\mu\nabla_\beta u_\alpha + \nabla_\alpha u^\mu\nabla_\mu u_\beta - \nabla_\beta u^\mu\nabla_\mu u_\alpha \\ &\quad + \frac{p+\varrho}{\nu^2}\left(\frac{\partial p}{\partial\varrho^2}\right)_s \nabla_\mu u^\mu[u_\alpha u^\lambda\nabla_\lambda u_\beta - u_\beta u^\lambda\nabla_\lambda u_\alpha] \\ &\quad + \left[\frac{\partial\nu^2}{\partial s} - \frac{1}{p+\varrho}\left(1+\frac{r}{\nu^2}\frac{\partial\nu^2}{\partial r}\right)\frac{\partial p}{\partial s} + \frac{1}{\varrho+p}\nu^2\frac{\partial\varrho}{\partial s}\right] \\ &\quad \times (u_\alpha\nabla_\mu u^\mu s_\beta - u_\beta\nabla_\mu u^\mu s_\alpha) \\ &\quad + \frac{1}{p+\varrho}\left(\frac{\partial\varrho}{\partial s} - \frac{1}{\nu^2}\frac{\partial p}{\partial s}\right)(s_\alpha u^\mu\nabla_\mu u_\beta - s_\beta u^\mu\nabla_\mu u_\beta).\end{aligned}$$

Combining the quantities in the system (3.14) with the above expression, we obtain that in fluid source gauge equations (3.29c) and (3.29d) correspond to

$$\nu^2 d^{\bar{\mu}}_{0\bar{\alpha}\bar{\mu}} - \frac{1}{p+\varrho}\nabla_{[0}q_{\bar{\alpha}]} = 0$$

and

$$\nu^2 d^{0}_{\bar{\alpha}0\bar{\beta}} + \frac{\nu^2}{p+\varrho}\nabla_{[\bar{\alpha}}q_{\bar{\beta}]} = 0,$$

respectively.

Next, notice that Eqs. (3.29g), (3.29h) and (3.29j) correspond to (2.5), (2.15) and (2.8) when written in fluid source gauge, whereas (3.29i) is the same as $\mathcal{L}_u\nabla_\alpha s = 0$, which is implied by Eqs. (3.14) ($\mathcal{L}$ is the Lie derivative).

Finally, we have the decomposition

$$\nabla^\mu T_{\mu\alpha} = w u_\alpha + q_\alpha$$

where w is given by the left-hand side of (2.5). From (3.10) and (3.13), we obtain

$$-\frac{1}{2}\mathscr{K} q_\alpha = \pi^\sigma_\mu \pi^\tau_\nu \pi^\lambda_\alpha F_{\sigma\tau\lambda}\pi^{\mu\nu} - \pi^\lambda_\alpha F_{\mu\lambda\nu}u^\mu u^\nu,$$

and

$$\frac{1}{2}\mathscr{K} w = \pi^\sigma_\mu \pi^\lambda_\nu F_{\sigma\tau\lambda}u^\tau \pi^{\mu\nu}.$$

With the help of these expressions and (3.17) and defining $\overline{\nabla}$ as in (3.28), it follows that

$$\begin{aligned}
&\pi^\mu_{(\alpha}\pi^\nu_{\beta)}F_{\mu\tau\nu}u^\tau - \frac{1}{3}\pi_{\alpha\beta}\pi^{\mu\nu}\pi^\sigma_\mu\pi^\lambda_\nu F_{\sigma\xi\lambda}u^\xi \\
&\quad = u^\mu \nabla_\mu E_{\alpha\beta} + E_{\mu\beta}\nabla_\alpha u^\mu \\
&\qquad + E_{\mu\alpha}\nabla_\beta u^\mu + \overline{\nabla}_\mu B_{\nu(\alpha}\varepsilon^{\mu\nu}_{\beta)} - 2u^\lambda \nabla_\lambda u_\mu \varepsilon^{\mu\nu}_{(\alpha}B_{\beta)\nu} - 3\pi^\mu_{(\alpha}\nabla_{|\mu|}u^\lambda E_{\beta)\lambda} \\
&\qquad - 2\pi^\mu_\lambda \nabla^\lambda u_{(\alpha}E_{\beta)\mu} + \pi_{\alpha\beta}\pi^{\mu\lambda}\nabla_\lambda u^\nu E_{\mu\nu} + 2\pi^{\mu\nu}\pi^\lambda_\mu \nabla_\lambda u_\nu E_{\alpha\beta} \\
&\qquad + \frac{1}{2}\mathscr{K}(p+\varrho)\left(\pi^\lambda_{(\alpha}\nabla_{|\lambda|}u_{\beta)} - \frac{1}{3}\pi^{\mu\nu}\pi^\lambda_\mu \nabla_\lambda u_\nu \pi_{\alpha\beta}\right),
\end{aligned}$$

which, in fluid source gauge, gives (3.29e). Equation (3.29f) is obtained by a similar argument after computing

$$\varepsilon^{\mu\nu}_{(\alpha}\pi^\lambda_{\beta)}\pi^\sigma_\mu\pi^\tau_\nu F_{\lambda\sigma\tau}.$$

References

[1] A. Anderson, Y. Choquet-Bruhat and J. W. York, Einstein–Bianchi hyperbolic system for general relativity, *Topol. Methods Nonlinear Anal.* **10** (1997) 353–373.

[2] A. M. Anile, *Relativistic Fluids and Magneto-Fluids* (Cambridge University Press, 1989).

[3] U. Brauer and L. Karp, Well-posedness of the Einstein–Euler system in asymptotically flat spacetimes: The constraint equations, *J. Differential Equations* **251** (2011) 1428–1446.

[4] U. Brauer and L. Karp, Well-posedness of the Einstein–Euler system in asymptotically flat spacetimes (2009), arXiv:0810.5045v2 [math.AP].

[5] U. Brauer and L. Karp, Local existence of solutions of self gravitating relativistic perfect fluids, *Comm. Math. Phys.* **32** (2014) 105–141; arXiv:1112.2405 [math.AP].

[6] U. Brauer and L. Karp, Local existence of classical solutions for the Einstein–Euler system using weighted Sobolev spaces of fractional order, *C. R. Acad. Sci. Paris, Ser. I* **345** (2007) 49–54.

[7] T. Carleman, Über eine isoperimetrisehe Aufgabe und ihre physikalisehen Anwendungen, *Math. Z.* **3** (1919) 1–7.

[8] Y. Choquet-Bruhat, *General Relativity and the Einstein Equations* (Oxford University Press, 2009).

[9] Y. Fourès-Bruhat (Y. Choquet-Bruhat), Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires, *Acta Math.* **88** (1952) 141–225.

[10] Y. Choquet-Bruhat, Théorèmes d'Existence en Mécanique des Fluides Relativistes, *Bull. Soc. Math. France* **86** (1958) 155–175.

[11] Y. Choquet-Bruhat and J. W. York, On H. Friedrich's formulation of the Einstein equations with fluid sources, *Topol. Methods Nonlinear Anal.* **18** (2001) 321–235.

[12] D. Christodoulou and S. Klainerman, *The Global Nonlinear Stability of the Minkowski Space* (Princeton University Press, 1993).

[13] P. Chrusciel, G. Galloway and D. Pollack, Mathematical general relativity: A sampler, *Bull. Amer. Math. Soc.* (*N.S.*) **47** (2010) 567–638; arXiv:1004.1016v2.

[14] J. Corvino and D. Pollack, Scalar curvature and the Einstein constraint equations, in *Surveys in Geometric Analysis and Relativity*, Adv. Lect. Math., Vol. 20 (Int. Press, 2011), pp. 145–188; arXiv:1102.5050v1 [math.DG].

[15] D. Coutand and S. Shkoller, Well-posedness of the free-surface incompressible Euler equations with or without surface tension, *J. Amer. Math. Soc.* **20** (2007) 829–930; arXiv:math/0511236v3 [math.AP].

[16] S. Dain and G. Nagy, Initial data for fluid bodies in general relativity, *Phys. Rev. D* **65** (2002) 084020, 15pp.

[17] T. Damour, General relativity today, in *Gravitation and Experiment*, Prog. Math. Phys., Vol. 52 (Birkhäuser, 2007), pp. 1–49; arXiv:0704.0754v1.

[18] M. M. Disconzi and D. G. Ebin, On the limit of large surface tension for a fluid motion with free boundary, *Comm. Partial Differential Equations* **39** (2014) 740–779; arXiv:1301.7507 [math.AP].

[19] M. M. Disconzi and V. P. Pingali, On the Choquet–Bruhat–York–Friedrich formulation of the Einstein–Euler equations, *Mod. Phys. Lett. A* **29** (2014) 1450205; arXiv: 1304.5739 [math.AP].

[20] D. G. Ebin, The equations of motion of a perfect fluid with free boundary are not well posed, *Comm. Partial Differential Equations* **12**(10) (1987) 1175–1201.

[21] G. Dixon, *Special Relativity, the Foundation of Macroscopic Physics* (Cambridge University Press, 1978).

[22] A. E. Fischer and J. E. Marsden, The Einstein evolution equations as a first-order quasi-linear symmetric hyperbolic system, I, *Comm. Math. Phys.* **28** (1972) 1–38.

[23] H. Friedrich, Evolution equations for gravitating ideal fluid bodies in general relativity, *Phys. Rev. D* **57** (1998) 2317–2322.

[24] H. Friedrich, Hyperbolic reductions for Einstein's equations, *Class. Quantum Grav.* **13** (1996) 1451–1469.

[25] H. Friedrich, On the hyperbolicity of Einstein's and other gauge field equations, *Comm. Math. Phys.* **100** (1985) 525–543.

[26] H. Friedrich, On the regular and the asymptotic characteristic initial value problem for Einsteins vacuum field equations, *Proc. Roy. Soc. London Ser. A* **375** (1981) 169–184.

[27] H. Friedrich, The asymptotic characteristic initial value problem for Einstein's vacuum field equations as an initial value problem for a first-order quasilinear symmetric hyperbolic system, *Proc. Roy. Soc. London Ser. A* **378** (1981) 401–421.

[28] H. Friedrich, On the global existence and the asymptotic behaviour of solutions to the Einstein–Maxwell–Yang–Mills equations, *J. Differential Geom.* **34** (1991) 275–345.

[29] H. Friedrich and G. Nagy, The initial boundary value problem for Einstein's vacuum field equation, *Comm. Math. Phys.* **201** (1999) 619–655.

[30] H. Friedrich and A. Rendall, The Cauchy problem for the Einstein equations, in *Einstein's Field Equations and Their Physical Implications* (Springer, 2000), pp. 127–223.

[31] J. Groah, J. Smoller and B. Temple, *Shock Wave Interactions in General Relativity*, A locally inertial Glimm scheme for spherically symmetric spacetimes, Springer Monographs in Mathematics (Springer, 2007).

[32] S. Hawking and G. Ellis, *The Large Structure of Space-Time*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 1973).
[33] J. M. Heinzle, N. Rohr and C. Uggla, Dynamical systems approach to relativistic spherically symmetric static perfect fluid models, *Class. Quantum Grav.* **20** (2003) 4567–4586.
[34] T. Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems, *Arch. Ration. Mech. Anal.* **58** (1975) 181–205.
[35] S. Kind and J. Ehlers, Initial-boundary value problem for the spherically symmetric Einstein equations for a perfect fluid, *Class. Quantum Grav.* **10** (1993) 2123–2136.
[36] A. Lichnerowicz, *Relativistic Hydrodynamics and Magnetohydrodynamics* (W. A. Benjamin, Inc., 1967).
[37] A. Lichnerowicz, L'intégration des équations de la gravitation relativiste et le problème des n corps, *J. Math. Pures Appl.* (9) **23** (1944) 37–63.
[38] L. Lichtenstein, Über einige Eigenschaftender Gleichgewichtsfiguren rotierender homogener Flüssigkeiten, deren Teilchen einander nach dem Newtonschen Gesetz anziehen, *Sitzungsberichte der Preussischen Akademie der Wissenschaften* (1918) 1120–1135.
[39] L. Lichtenstein, *Gleichgewichtsjiguren Rotierender Flussigkeiten* (Springer, 1933).
[40] L. Lindblom, Static uniform-density stars must be spherical in general relativity, *J. Math. Phys.* **29** (1988) 436–439.
[41] A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables* (Springer-Verlag, 1984).
[42] T. Makino, On a local existence theorem for the evolution equation of gaseous stars, in *Patterns and Wuues* (North-Holland, 1986), pp. 459–479.
[43] T. Makino, On spherically symmetric stellar models in general relativity, *J. Math. Kyoto Univ.* **38** (1998) 55–69.
[44] E. T. Newman and R. Penrose, An approach to gravitational radiation by a method of spin coefficients, *J. Math. Phys.* **3** (1992) 566–578.
[45] R. Penrose, Zero rest-mass fields including gravitation: Asymptotic behaviour, *Proc. Roy. Soc. London Ser. A* **284** (1965) 159–203.
[46] D. Pugliese and J. A. V. Kroon, On the evolution equations for ideal magnetohydrodynamics in curved spacetime, to appear in *Gen. Relat. Gravit.*; DOI: 10.1007/s10714-012-1424-6. arXiv:1112.1525 [gr-qc].
[47] A. D. Rendall, Prospects for a theory of general relativistic fluid bodies, http://gravityresearchfoundation.org /pdf/awarded/1991/rendall.pdf.
[48] A. D. Rendall, The initial value problem for a class of general relativistic fluid bodies, *J. Math. Phys.* **33** (1992) 1047–1053.
[49] A. D. Rendall and B. G. Schmidt, Existence and properties of spherically symmetric static fluid bodies with a given equation of state, *Class. Quantum Grav.* **8** (1991) 985–1000.
[50] R. K. Sachs, Gravitational waves in general relativity: VI. The outgoing radiation condition, *Proc. Roy. Soc. London Ser. A* **264** (1961) 309–338.
[51] B. G. Schmidt, Conditions on a connection to be a metric connection, *Comm. Math. Phys.* **29** (1973) 55–59.
[52] B. Schmidt (ed.), *Einsteins Field Equations and Their Physical Implications* (Springer, 2000).
[53] P. Secchi, On the uniqueness of motion of viscous gaseous stars, *Math. Methods Appl. Sci.* **13** (1990) 391–404.
[54] P. Secchi, On the motion of gaseous stars in the presence of radiation, *Comm. Partial Differential Equations* **15** (1990) 185–204.

[55] P. Secchi, On the evolution equations of viscous gaseous stars, *Ann. Scuola Norm. Sup. Pisa* **36** (1991) 295–318.
[56] L. S. Shapiro and S. A. Teukolsky, *Black Holes*, *White Dwarfs and Neutron Stars* (Wiley, 1983).
[57] J. Shatah and C. Zeng, Geometry and *a priori* estimates for free boundary problems of the Euler's equation, *Comm. Pure Appl. Math.* **61** (2008) 698–744.
[58] M. E. Taylor, *Pseudodifferential Operators and Nonlinear PDE* (Birkhäuser, 1991).
[59] A. H. Taub, On circulation in relativistic hydrodynamics, *Arch. Ration. Mech. Anal.* **3** (1959) 312–329.
[60] H. Triebel, *Interpolation Theory*, *Function Spaces*, *Differential Operators* (North Holland, 1978).
[61] S. Weinberg, *Gravitation and Cosmology*: *Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, Inc., 1972).
[62] S. Weinberg, *Cosmology* (Oxford University Press, 2008).
[63] J. W. York, Covariant decompositions of symmetric tensors in the theory of gravitation. *Ann. Inst. H. Poincaré Sect. A* (*N.S.*) **21** (1974) 319–332.
[64] Y. Zel'dovich and I. D. Novikov, *Relativistic Astrophysics*, *Vol.* 1: *Stars and Relativity* (Chicago University Press, 1971).