3. The LSZ reduction formula

We want to find an useful formula to compute scattering amplitudes. In the free theory we can create a state of one particle by acting with $a^\dagger(\vec{k})$ on the vacuum:

$$|k\rangle = a^\dagger(\vec{k})|0\rangle$$

Of course, $a(\vec{k})$ kills the vacuum $a(\vec{k})|0\rangle = 0$

And we can assume that $|0\rangle$ is normalized: $\langle 0|0\rangle = 1$.

Also notice that $\langle k|k'\rangle = \langle 0|a(\vec{k})\, a^\dagger(\vec{k}')|0\rangle =$

$$= \langle 0|\underbrace{[a(\vec{k}), a^\dagger(\vec{k}')]}_{=(2\pi)^3 2\omega(\vec{k})\delta(\vec{k}-\vec{k}')}|0\rangle + \langle 0|a^\dagger(\vec{k}')\underbrace{a(\vec{k})|0\rangle}_{=0} = (2\pi)^3 2\omega\, \delta(\vec{k}-\vec{k}')$$

which simply means that in the free theory a state $|k\rangle$ can not evolve to a state $|k'\rangle$ if $k \neq k'$

Now let us turn our attention to the scattering process. How do we model a scattering? Basically we think of it as follows. At the very distant past ~~we start with~~ i.e, $t=-\infty$, we start with n free particles with momenta $k_1,\dots,k_n$. This is described by a state $|i\rangle$. Notice that since the particles are free at $t=-\infty$ then they don't interact. (This is reasonable because particles behave essentially as free particles if they are sufficiently isolated, and in a scattering process particles start isolated from each othe and only then they come together to "collide" (i.e, scatter)). Then this state this state evolves and at finite time the particles are no longer isolated, so they interact. The interaction can create or destroy particles so after the interaction we have m particles with momenta $\tilde{k}_1,\dots,\tilde{k}_m$. The system evolves and in the distant future the particles separate apart from each other, becoming isolated — what means that when $t = +\infty$ the state is again described by free particles; denote this final state $|f\rangle$.

So, our basic question is: starting with a system of, say, two particles with momenta k_1, k_2, what is the probability of the scattering to produce, say, three particles with momenta $\tilde{k}_1, \tilde{k}_2, \tilde{k}_3$? If the the initial state is $|i\rangle$ and the ~~final~~ final state $|f\rangle$, we basically are asking $\langle f | i \rangle$ (of course, this is the amplitude, the probability is $|\langle f | i \rangle|^2$ ~~…~~ St 52).

Consider creation and annihilation operators for the the interacting theory now. The first thing to notice is that these operators are no ~~long~~ longer time independent as in the free theory ~~…~~

Indeed, for an interaction Lagrangian of the form $\mathcal{L}_I = \lambda \varphi^{n+1}$ (which is all we consider) we have

$$\Box \varphi + m^2 \varphi + \lambda \varphi^n = 0$$

(where we absorb the $n+1$ coming from the derivative on λ)

Assuming we can write a Fourier representation of the solution, we have:

$$\varphi = \int \left(a(k) e^{-ikx} + a^*(k) e^{ikx} \right) d\vec{k}$$

, so

+ cross terms involving $\partial_t a$. We'll show that $\partial_t a \equiv 0 \Rightarrow \varphi \equiv 0$, that's why such terms are not written

$$\Box \varphi = \int (\Box a(k)) e^{-ikx} + (\Box a^*(k)) e^{ikx} + a(k) \Box e^{-ikx} + a^*(k) \Box e^{ikx}$$

But $\Box = e^{-ikx} = (\partial_t^2 - \Delta) e^{-ikx} = \left((-i\omega)^2 - (-i\vec{k})^2 \right) e^{-ikx} =$

$= (-\omega^2 + \vec{k}^2) e^{-ikx} = -m^2 e^{-ikx}$; analogous for $\Box e^{ikx}$, so

$$\Box \varphi = \int \left(\Box a(k) e^{-ikx} + \Box a^*(k) e^{ikx} \right) - m^2 \underbrace{\int a(k) e^{-ik} + a^*(k) e^{ikx}}_{= \varphi}$$

and so $\Box \varphi + m^2 \varphi = \int \left(\Box a(k) e^{-ikx} + \Box a^*(k) e^{ikx} \right) d\vec{k}$.

Since $a(k)$ doesn't depend on x $\Box a = \partial_t^2 a$, so if a does not depend on t we have $\Box \varphi + m^2 \varphi = 0$ and so

$\Box \varphi + m^2 \varphi + \lambda \varphi^n = 0 \implies \varphi^n = 0$, hence $a = a(t, k)$

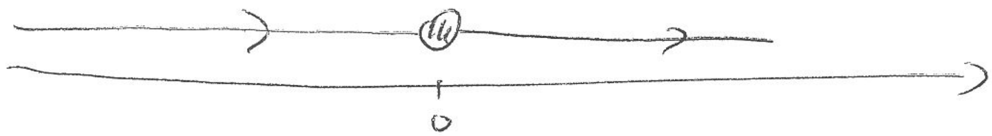
Notice that the Fourier transf. is spatial only, so we don't get rid of t)

Another important point here is the following: how do we know that $a^\dagger(\vec{k},t)$ and $a(\vec{k},t)$ are indeed creation and annihilation operators for the interacting theory. The answer is that we actually don't know! All we know is that they are Fourier modes for the interacting field — and then by analogy we would like to ~~sa~~ say they are creation/annihilation operators.

However, we will see that is reasonable to suppose that the particles are free in the very distant past/future so that ~~then~~ ~~means~~ we may naturally assume that the modes $a(\vec{k},t)$ go to $a(\vec{k})$ — the operators of the free theory, which in turn are ~~now~~ actually creation/annihilation — when $t \to \pm\infty$.

we write $a^\dagger(k) = a^\dagger(\vec{k}, t)$, $a(k) = a(\vec{k}, t)$.

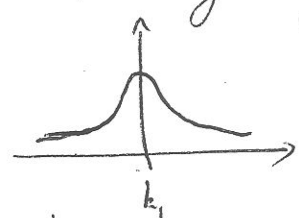
Let us suppose now that we study the scattering where we start with two particles with momenta k_1 and k_2 and end up with two particles with momenta $\bar{k}_1$ and $\bar{k}_2$ (the generalization to more particles will be straightforward). The problem here is that it is not true that $|i\rangle = a^\dagger(\vec{k}_1, t)\, a^\dagger(\vec{k}_2, t)\,|0\rangle$ since $|i\rangle$ is supposed to be a state of free particles and a's are not creation operators of the free theory. To overcome this difficulty, let us consider the following reasoning. First, suppose that the frame which we are using is such that the scattering happens at the spatial origin, at $t = -\infty$ the particles are at $x = -\infty$ (where they are isolated and hence free) and at $t = +\infty$ they are at $x = +\infty$ (free again)



How do we describe a particle which have momentum near $\vec{k}_1$ when it comes near the origin and is isolated at $x = -\infty$? For *free* particles we can consider the operator:

$$a_1^{+} = \int d\vec{k}\, f_1(\vec{k})\, a^{+}(\vec{k})$$

where $$f_1(\vec{k}) \propto e^{-\frac{(\vec{k}-k_1)^2}{4\sigma^2}}$$, so a_1 is an operator which creates ~~an~~ a wave package of width σ and localized near $\vec{k}_1$ and localized in position space near the origin.



Consider then the state $a_1^{+}|0\rangle$. When time evolves, this state (in Schrödinger picture) will propagate and spread out. Hence at $t = \pm\infty$ the package will be totally spread in momentum space, what means that the particle will be localized in position space. If we consider more than one particle, then the same reasoning shows that the state $a_1^{+} a_2^{+}|0\rangle$, $\vec{k}_1 \neq \vec{k}_2$, has two particles widly separated in the distant past.

If we do the same for the operators of the interacting theory, i.e:

$$a_1^\dagger(t) = a_1^\dagger(\vec{k}_1, t) = \int d\vec{k}\, f_1(\vec{k})\, a^\dagger(\vec{k}, t)$$

then our guess for $|i\rangle$ would be:

$$|i\rangle = \lim_{t\to-\infty} a_1^\dagger(t)\, a_2^\dagger(t)\, |0\rangle \quad , \quad \vec{k}_1 \neq \vec{k}_2$$

This seemingly suspect assumption, i.e., that we can ~~ap~~ apply the previous reasoning to the time dependent operators of the interacting theory, can be justified on physical grounds, see (Sre, 51ff). We can normalize the state $\langle i|i\rangle = 1$ by appropriately normalizing the wave packages (Sre, 53). Analogously:

$$|f\rangle = \lim_{t\to\infty} a_{\tilde{1}}^\dagger(t)\, a_{\tilde{2}}^\dagger(t)\, |0\rangle \quad , \quad \vec{k}_1 \neq \vec{k}_2$$

and $\langle f|f\rangle = 1$.

Now we need to find an useful formula for $\langle f|i\rangle$. First use the fundamental theorem of calculus:

$$a_1^\dagger(+\infty) - a_1^\dagger(-\infty) = \int_{-\infty}^{+\infty} dt\ \partial_0 a_1^\dagger(t)$$

notice that we are using that $\partial_0 a\Big|_{-\infty}^{+\infty} = 0$ as a is free (time indep) at $\pm\infty$ (see p. 17)

Now $a_1^\dagger(t) = \int d\vec{k}\ f_1(\vec{k})\ a^\dagger(\vec{k},t)$ and

$$a^\dagger(\vec{k},t) = -i\int d\vec{x}\ e^{-ikx}\overleftrightarrow{\partial_0}\varphi(x)$$

(recall that now φ is no longer a free field), so

$$a_1^\dagger(+\infty) - a_1^\dagger(-\infty) = -i\int d\vec{k}\ f_1(\vec{k})\int dx\ \partial_0\left(e^{-ikx}\overleftrightarrow{\partial_0}\varphi(x)\right)$$

where $\int dt\int d\vec{x} = \int dx$

Now: $\partial_0\left(e^{-ikx}\overleftrightarrow{\partial_0}\varphi(x)\right) = \partial_0\left(e^{-ikx}\partial_0\varphi(x) - (-i\omega)e^{-ikx}\varphi(x)\right)$

this comes from the formula for $\overleftrightarrow{\partial}$

$$= \cancel{-i\omega e^{-ikx}\partial_0\varphi(x)} + e^{-ikx}\partial_0^2\varphi(x) + \cancel{i\omega e^{-ikx}\partial_0\varphi(x)}$$

$$+ i\omega(-i\omega)e^{-ikx}\varphi = e^{-ikx}(\partial_0^2 + \omega^2)\varphi(x)$$

$$= e^{-ikx}(\partial_0^2 + \vec{k}^2 + m^2)\varphi(x)$$

using $\omega = \sqrt{\vec{k}^2 + m^2}$

So $a_1^\dagger(+\infty) - a_1^\dagger(-\infty) = -i \int d\vec{k}\, f_1(\vec{k}) \int dx\, e^{-ikx} \left(\partial_0^2 + \vec{k}^2 + m^2\right) \varphi(x)$

Now notice that

$$\int dx\, e^{-ikx}\, \vec{k}^2\, \varphi(x) = - \int dx \left(\Delta e^{-ikx}\right) \varphi(x)$$

$$= - \int dx\, e^{-ikx}\, \Delta \varphi(x)$$

where Δ is only spatial Laplacian and in the last step we integrated by parts. So

$$a_1^\dagger(+\infty) - a_1^\dagger(-\infty) = -i \int d\vec{k}\, f_1(\vec{k}) \int dx\, e^{-ikx} \left(\partial_0^2 - \Delta + m^2\right) \varphi(x)$$

$$= -i \int d\vec{k}\, f_1(\vec{k}) \int dx\, e^{-ikx} \left(\Box + m^2\right) \varphi(x).$$

Notice that for a free theory $(\Box + m^2)\varphi = 0$ and we get (as expected) $a_1^\dagger(+\infty) = a_1^\dagger(-\infty)$. For an interacting theory with interaction term $\mathcal{L}_I$ (e.g. $\mathcal{L}_I = \frac{\lambda}{4!}\varphi^4$) then

$$(\Box + m^2)\varphi = \frac{\delta \mathcal{L}_I}{\delta \varphi} \quad \left(\text{e.g.}, = \frac{\lambda}{6}\varphi^3\right) \neq 0 .$$

We have an analogous equation for the annihilation operators:

$$a_{\tilde{1}}(+\infty) - a_{\tilde{1}}(-\infty) = i \int d\vec{k}\, f_{\tilde{1}}(\vec{k}) \int dx\, e^{ikx} (\Box + m^2)\, \varphi(x)$$

Returning to the amplitude we have:

$$\langle f | i \rangle = \langle 0 | a_{\tilde{1}}(+\infty)\, a_{\tilde{2}}(+\infty)\, a_1^\dagger(-\infty)\, a_2^\dagger(-\infty) | 0 \rangle$$

Since the operators appear in time order (recall that $a_1^\dagger, a_2^\dagger$ commute bc $k_1 \neq k_2$, analogous $a_{\tilde{1}}, a_{\tilde{2}}$) we can also write

$$\langle f | i \rangle = \langle 0 | T\left(a_{\tilde{1}}(+\infty)\, a_{\tilde{2}}(+\infty)\, a_1^\dagger(-\infty)\, a_2^\dagger(-\infty)\right) | 0 \rangle$$

Using the previous expressions for $a, a^\dagger$:

$$= \langle 0 | T\Bigg(\left[a_{\tilde{1}}(-\infty) + i \int\!\!\int e^{ikx}\right] \left[a_{\tilde{2}}(-\infty) + i \int\!\!\int e^{ikx}\right] \left[a_1^\dagger(+\infty) + i \int\!\!\int e^{-ikx}\right]$$

$$\left[a_2^\dagger(+\infty) + i \int\!\!\int e^{-ikx}\right] | 0 \rangle$$

The time ordering operator will move all the $a(-\infty)$ to the right and $a^\dagger(+\infty)$ to the left; on the right $a(-\infty)$ will act on $|0\rangle$ giving zero. Therefore only the terms with integral remain:

$$= \langle 0| T \Big(i^2 \int d\vec{k}\, f_{\tilde{1}}(\vec{k}) \int dx\, e^{ikx} (\Box + m^2)\, \varphi(x) \times$$

$$\int d\vec{k}\, f_{\tilde{2}}(\vec{k}) \int dx\, e^{ikx} (\Box + m^2)\, \varphi(x) \times$$

$$i^2 \int d\vec{k}\, f_1(\vec{k}) \int dx\, e^{-ikx} (\Box + m^2)\, \varphi(x) \times$$

$$\int d\vec{k}\, f_2(\vec{k}) \int dx\, e^{-ikx} (\Box + m^2)\, \varphi(x) \Big) |0\rangle$$

Now we don't need the wave packages any more, so we take the limit $\sigma \to 0$ so that $f_1(\vec{k}) = \delta(\vec{k} - \vec{k}_1)$, $f_2(\vec{k}) = \delta(\vec{k} - \vec{k}_2)$, $f_{\tilde{1}}(\vec{k}) = \delta(\vec{k} - \vec{k}_1)$ etc. Then the delta functions kill the integrals in $\vec{k}$ and we have:

$$\langle f | i \rangle = i^{2+2} \langle 0 | T \Big(\int dx\, e^{i\tilde{k}_1 x} (\Box + m^2)\varphi(x) \int dx\, e^{i\tilde{k}_2 x} (\Box + m^2)\varphi(x)$$

$$\int dx\, e^{-ik_1 x} (\Box + m^2)\varphi(x) \int dx\, e^{-ik_2 x} (\Box + m^2)\varphi(x) \Big) | 0 \rangle =$$

all x's are dummy

$$i^{2+2} \langle 0 | T \int d\tilde{x}_1\, e^{i\tilde{k}_1 \tilde{x}_1} (\Box_{\tilde{x}_1} + m^2)\varphi(\tilde{x}_1) \int d\tilde{x}_2\, e^{i\tilde{k}_2 \tilde{x}_2} (\Box_{\tilde{x}_2} + m^2)\varphi(\tilde{x}_2)$$

$$\int dx_1\, e^{-ik_1 x_1} (\Box_{x_1} + m^2)\varphi(x_1) \int dx_2\, e^{-ik_2 x_2} (\Box_{x_2} + m^2)\varphi(x_2) \Big) | 0 \rangle$$

$$= i^{2+2} \int d\tilde{x}_1\, d\tilde{x}_2\, dx_1\, dx_2\, e^{i\tilde{k}_1 \tilde{x}_1} (\Box_{\tilde{x}_1} + m^2)\, e^{i\tilde{k}_2 \tilde{x}_2} (\Box_{\tilde{x}_2} + m^2)$$

$$e^{-ik_1 x_1} (\Box_{x_1} + m^2)\, e^{-ik_2 x_2} (\Box_{x_2} + m^2) \langle 0 | T(\varphi(\tilde{x}_1)\varphi(\tilde{x}_2)\varphi(x_1)$$

$$\varphi(x_2)) | 0 \rangle$$

This is the <u>reduction formula</u>. It tells us that if we know the correlation functions $\langle 0 | T(\varphi(\tilde{x}_1) \cdots \varphi(x_2)) | 0 \rangle$ then we can explicitly compute the scattering amplitude.