6. A first look ~~at~~ Feynman diagrams

As we saw on the previous section, the computation of correlation functions involves a lot of combinatorics. An efficient way of organizing this combinatorics is given by Feynman diagrams — which also helps ~~as~~ us with the interpretation of the physical process.

We saw that the two point function of the φ^4-theory has order λ term:

$$-\frac{\lambda}{2}\,\Delta(0)\int \Delta(x_1-z)\,\Delta(x_2-z)\,dz$$

, so the two point function can be written

$$G(x_1,x_2) = i\,\Delta(x_1-x_2) - \frac{\lambda}{2}\,\Delta(0)\int \Delta(x_1-z)\,\Delta(x_2-z)\,dz + O(\lambda^2)$$

the term $i\Delta(x_1-x_2)$ correspond to order zero — i.e, no interaction and hence is the propagator of the free theory that we computed before.

Use the same ideas we can compute the four point function (Ry p. 205), up to order λ:

$$G(x_1, x_2, x_3, x_4) = -\Big(\Delta(x_1-x_2)\,\Delta(x_2-x_3) + \Delta(x_1-x_3)\,\Delta(x_2-x_4) + \Delta(x_1-x_4)\,\Delta(x_2-\cancel{x_3})\Big) - \frac{i\lambda}{4!}\, 24 \int \Delta(x_1-z)\,\Delta(x_2-z)\,\Delta(x_3-z)\,\Delta(x_4-z)\; dz$$

Here we are omiting an extra term in λ which, as we'll see, correspond to a disconnect diagram and does not contribute to the scattering.

Although it must be included to compute the complete $G(x_1,x_2,x_3,x_4)$

a component that is disconnected from external vertices

We saw that the physical relevant quantities are the amplitudes, which can be obtained from the n-point functions via the reduction formula. However, the correlation functions themselves have nice physical interpretation.

First let us discuss the meaning of the propagator Δ. Recall that it satisfies (p. 41)

$$(\Box + m^2 - i\varepsilon)\,\Delta(x) = -\delta(x)$$

Physically $\Delta(x)$ describes the amplitude for a disturbance in the field to propagate from the origin to the spacetime point x (Ze 23) and, of course, $\Delta(x-y)$ is the amplitude for a disturbance to propagate from x to y. Therefore the n-point function has a nice interpretation, for example, $G(x_1, x_2, x_3, x_4)$ represents the following process: two particles (mesons = field excitations) are emitted from x_1 and x_2 and propagate to z with amplitude $\Delta(x_1 - z)\Delta(x_2 - z)$, they scatter with amplitude $-i\lambda$, and then they propagate, with amplitude $\Delta(x_3 - z)\Delta(x_4 - z)$ to x_3 and x_4, where they are (re-)absorbed by the field (Ze 51, St 86). The integration over z simply means that the interaction can happen at any spacetime point.

The sum of different order terms simply means adding amplitudes for different physical process, i.e., different intermediate process in the scattering from x_1, x_2 to x_3, x_4 (we'll see in a second that they indeed correspond to different intermediate steps).

Let's represent Δ in momentum space:

$$\Delta(x-z) = \int \frac{e^{\pm ik(x-z)}}{k^2 - m^2 + i\varepsilon} \frac{dk}{(2\pi)^4}$$

where we can choose $\pm$ by changing the dummy variable of integration. Plugging in the λ term of $G(x_1, x_2, x_3, x_4)$:

$$\int \Delta(x_1 - z) \cdots \Delta(x_4 - z)\, dz = \int\!\!\int\cdots \frac{e^{-ik_1x_1}}{k_1^2 - m^2 + i\varepsilon} \frac{e^{-ik_2x_2}}{k_2^2 - m^2 + i\varepsilon} \frac{e^{ik_3x_3}}{k_3^2 - m^2 + i\varepsilon}$$

$$\frac{e^{ik_4x_4}}{k_4^2 - m^2 + i\varepsilon} \underbrace{\left(\int e^{-i(k_1 + k_2 - k_3 - k_4)z}\, dz \right)}_{= (2\pi)^4 \delta(k_1 + k_2 - k_3 - k_4)} \frac{dk_1}{(2\pi)^4} \frac{dk_2}{(2\pi)^4} \frac{dk_3}{(2\pi)^4} \frac{dk_4}{(2\pi)^4}$$

Therefore, the fact that the interaction could happen anywhere in spacetime translate in conservation of momentum, i.e, interaction happens only then the sum of ingoing momentum equals the sum of outgoing momentum : $k_1 + k_2 = k_3 + k_4$. Also:

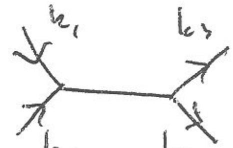
$$= \int \cdots \int e^{-ik_1} \cdots e^{ik_4 x_4} \underbrace{\left(\frac{1}{k_1^2 - m^2 + i\varepsilon} \cdots \frac{1}{k_4^2 - m^2 + i\varepsilon} (2\pi)^4 \delta(k_1 + k_2 - k_3 - k_4) \right)}_{G(k_1, k_2, k_3, k_4)} \frac{dk_1}{(2\pi)^4} \cdots \frac{dk_4}{(2\pi)^4}$$

and we identify the order λ term of $G(x_1, \ldots, x_4)$ as the Fourier transform of $G(k_1 \ldots k_4)$ which we interpret as the order λ term of the 4-point c.f. in momentum space. (Zee 53, St 95)

Notice that k_i are four vectors and conservation of momenta reads $\sum k_i = 0$; the use of negative signs for "outgoing" momenta is purely pictorical.

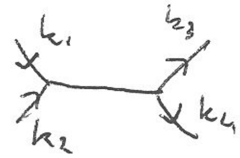
Looking carefully at how we construct the n-point function (including the combinatorics) and passing to momentum space, we can derive the following <u>Feynman rules</u> for writing directly $G(k_1, \ldots k_n)$. for φ^4 theory.

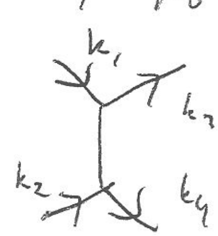
0. A $(n+m)$-point function corresponding to n incoming particles with momenta $k_1, \ldots, k_n$ and m outgoing particles with momenta $k_{n+1}, \ldots, k_{m+n}$, at order λ^p in perturbation theory, corresponds to diagrams corresponds to diagrams with $n+m$ external lines and p vertices, n external lines (corresponding to the incoming n particles) are drawn on the left and the m external lines are drawn on the right (on the bottom/top if we imagine time going from the bottom to the top). At each vertex exactly 4 lines meet (because of φ^4), we'll discuss the general case later).

1. Label the n incoming and m outgoing external edges according to their momenta. Incoming lines have an arrow pointing to the vertex and outgoing ones point out of the vertex, e.g., 

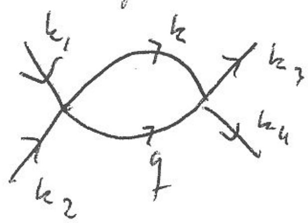
.

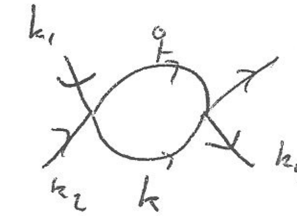
Identify all distinguishable/inequivalent connected graphs representing the process. (as we'll argue later disconnected graphs don't contribute to the n-point function). By distinguishable/inequivalent we mean the following: two graphs Γ_1 and Γ_2 are equivalent if there exists a map $g: \Gamma_1 \to \Gamma_2$ mapping edges to edges, vertices to vertices, preserving the incidence relation and fixing the external edges. (mathematically we are simply saying that graphs are equivalent if they are ~~[illegible]~~ related by an automorphism of labelled graphs, Po p. 7).

For example, 

and 

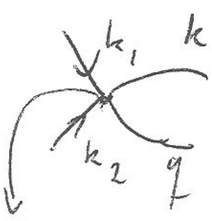
~~are~~ inequivalent: fixing the external edges there is no way of mapping one onto the other (notice that these graphs are not from φ^4 since they have 3 lines instead of 4 meeting at each vertex).

~~These~~ The graphs 

and 

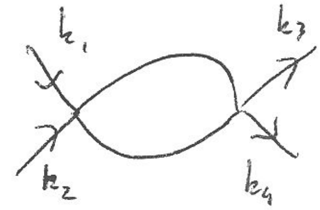
are equivalent since we can flip the loop 

without altering the external ~~ver~~ lines. Notice that the incidence relation is preserved by this flipping: on the first vertex, name it by u, we have k_1, k_2, q, k connected to it:

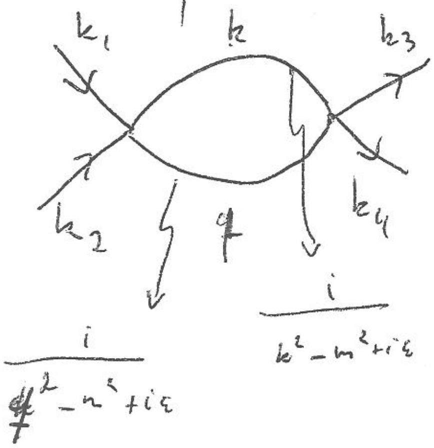


; then the flip g maps $k_1 \mapsto k_1$, $k_2 \mapsto k_2$, $u \mapsto u$, $k \mapsto q$, $q \mapsto k$, and on $g(u)$ the four lines which meet are $g(k_1)$, $g(k_2)$, $g(k)$, $g(q)$.

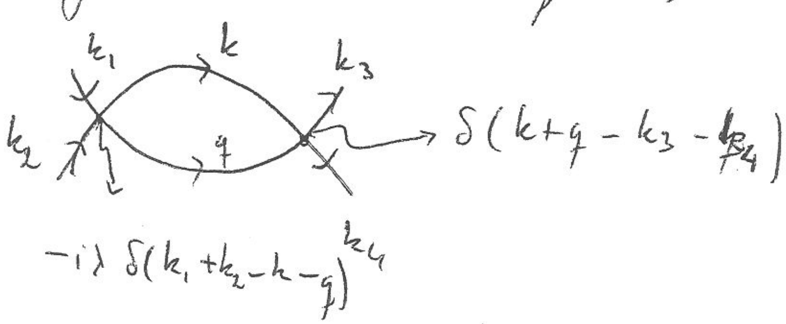
2. Label each line with a momentum k and associate with it the propagator $\dfrac{i}{k^2 - m^2 + i\varepsilon}$ (of course, the external lines are already labelled).

For example, 

is a 4 pt functions on order λ^2 (two vertices); we have



3. Associate with each interaction vertex the coupling $-i\lambda$ and $(2\pi)^4\,\delta\left(\sum_i k_i - \sum_j k_j\right)$ forcing the sum of the momenta ~~carrying~~ $\sum_i k_i$ coming into the vertex to be equal to the sum $\sum_j k_j$ of the momenta going out out of ~~the~~ vertex.

~~4~~. 

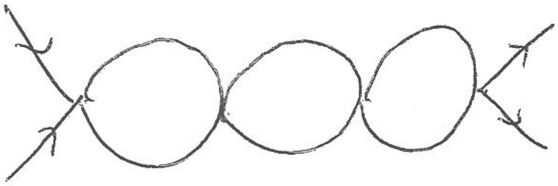
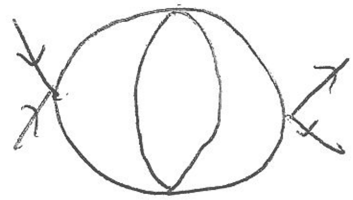
4. Momenta associated with internal lines are to be integrated over with the measure $\frac{dk}{(2\pi)^4}$

5. Finally multiply ~~those~~ each diagram by $\frac{1}{|\mathrm{Aut}\,\Gamma|}$, ie, one over the size of the symmetry group of the graph. So, we need to count how many maps g ~~ws~~ with the properties of item 1 exist. We'll see later a formula which gives part of the answer to $|\mathrm{Aut}\,\Gamma|$. ; $|\mathrm{Aut}\,\Gamma|$ (or sometimes $1/|\mathrm{Aut}\,\Gamma|$, is called symmetry factor. (Zee 53, Sr 87, St 88 and 96)

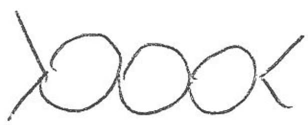
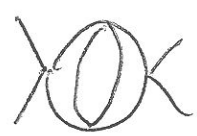
Of course, to compute the order λ^n term of G, which we'll denote by G_n or G_{λ^n} (so $G = G_0 + G_1 + \dots$) we need to add ~~these~~ over the inequivalent graphs corresponding to that order, ie,

$$G_n(k_1, \dots, k_\ell) = \sum_{\substack{\text{sum} \\ \text{of inequival} \\ \text{graphs with } n\text{-vertices} \\ \text{and } \ell \text{ external lines and} \\ 4 \text{ lines meeting at each vertex}}} (\text{graphs}).$$

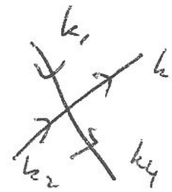
For example:

 and 

are both terms ~~unpref~~ occurring at order 4 (four vertices) of the 4-point function. They are inequivalent, so

$G_4(k_1,k_2,k_3,k_4) =$  $+$  $+$ other possible inequivalent term.

Let's first see how to compute some diagrams using the rules and after that we discuss some interpretations and consequences

At first order on the 4-point function the only (connected) diagram is 

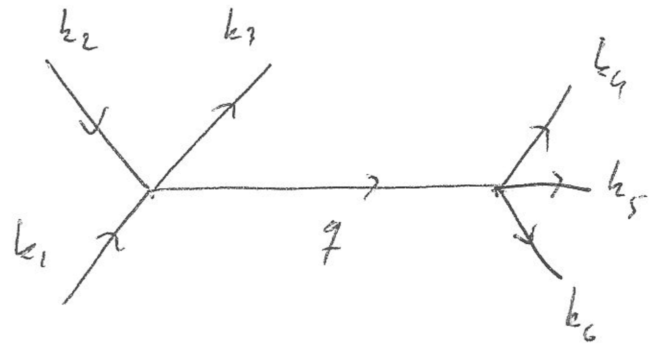
Applying the rules we find:

$$-i\lambda \frac{1}{k_1^2-m^2+i\epsilon} \cdots \frac{1}{k_4^2-m^2+i\epsilon}\, \delta(k_1+k_2-k_3-k_4)$$

Notice that we don't need to carry the term $\prod_{i=1}^{4} \frac{i}{k_i^2 - m^2 + i\epsilon}$ or, more generally $\prod_{i=1}^{n} \frac{i}{k_i^2 - m^2 + i\epsilon}$ since this term always appears. So we can add another rule:

6. Do not associate a propagator with external lines (more precisely, it is implicitly understood).

Let us look at another term, now on the 6-point function:

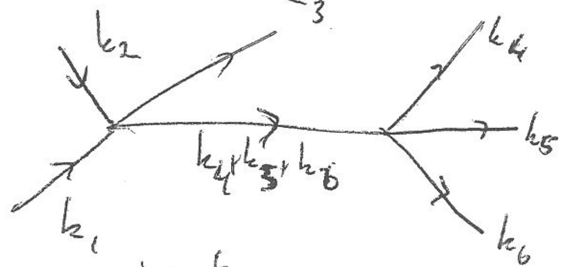


Notice that this term happens on second order on perturbation theory. Applying the rules:

$$(-i\lambda)^2 \int \frac{i}{q^2 - m^2 + i\epsilon} (2\pi)^4 \delta(k_1 + k_2 - k_3 - q)(2\pi)^4 \delta(q - (k_4 + k_5 + k_6))$$

$$\frac{dq}{(2\pi)^4} = (2\pi)^4 (-i\lambda)^2 \frac{i}{(k_4 + k_5 + k_6)^2 - m^2 + i\epsilon} \delta(k_1 + k_2 - k_3 - k_4 + k_5 - k_6)$$

From this example we see that we don't need to write the delta function to simply annihilate it on the next step. We could have
by integration
written directly



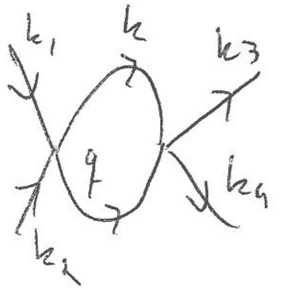
So, applying the rules 1–6 for this diagram we have the same answer quickly:

$$(-i\lambda)^2 \frac{i}{(k_4+k_5+k_6)^2-m^2+i\varepsilon}(2\pi)^4\delta(k_1+k_2-k_3-k_4-k_5-k_6).$$

Notice that overall momentum conservation delta function. If you ~~think~~ think a bit you'll conclude that after all the δ's are integrated over we'll end up with an overall ~~momentum~~ momentum conservation δ-function. So there is no need to carry this term either and we may add another rule:

7. A term $(2\pi)^4\delta(\sum_{in} k_i - \sum_{out} k_i)$ is implicitly understood.

Let us now look at the example



As we said, we know that we don't need to write one of the δ-functions to simply annihilate it by integration, so we could have written $q = k_1 + k_2 - k$. But let's do all steps one last time.

Applying the rules 1–6 (the factor $\frac{1}{2}$ is the symmetry factor)

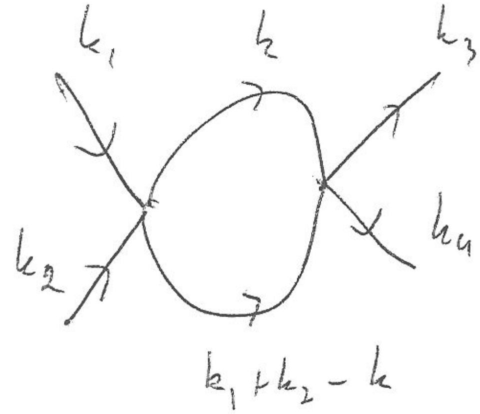
$$\frac{1}{2}(-i\lambda)^2 (2\pi)^4 (2\pi)^4 \int \delta(k_1+k_2-q-k)\,\delta(k+q-k_3-k_4)$$

$$\frac{i}{k^2-m^2+i\varepsilon}\;\frac{i}{q^2-m^2+i\varepsilon}\;\frac{dk}{(2\pi)^4}\;\frac{dq}{(2\pi)^4} = \frac{1}{2}(-i\lambda)^2 \int \delta(k+k_1+$$

$$k_2-k-k_3-k_4)\;\frac{i}{k^2-m^2+i\varepsilon}\;\frac{i}{(k_1+k_2-k)^2-m^2+i\varepsilon}\;\boxed{q = k_1+k_2-k}$$

$$= \frac{1}{2}(-i\lambda)^2\,\delta(k_1+k_2-k_3-k_4)\int \frac{i}{k^2-m^2+i\varepsilon}\;\frac{i}{(k_1+k_2-k)^2-m^2+i\varepsilon}\,dk$$

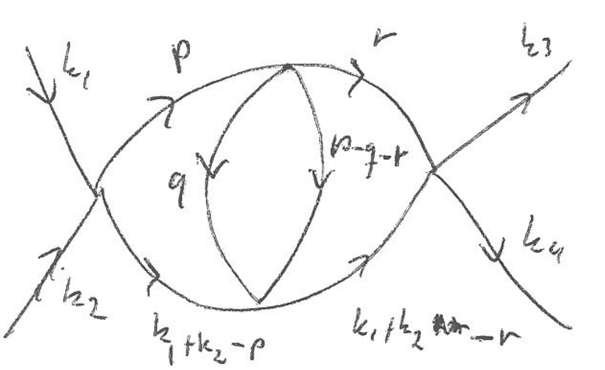
So, as we ~~said~~ said, we could've have written



. Now, if we apply the rules 1–7, noticing that now "integration over momenta associated with internal lines" on rule 4 is to be interpreted as "integration over momenta which are not fixed by conservation on each vertex" (which in this example is k) we get

$$\frac{1}{2}(-i\lambda)^2\int \frac{i}{k^2-m^2+i\varepsilon}\,\frac{i}{(k_1+k_2-k)^2-m^2+i\varepsilon}\,\frac{dk}{(2\pi)^4}$$

Let's look at one more example:

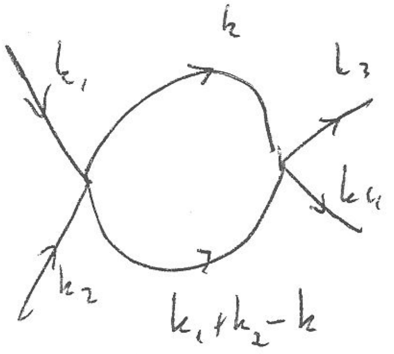


which happens at order 4 in perturbation theory

Applying 1–7 but omitting the symmetry factor:

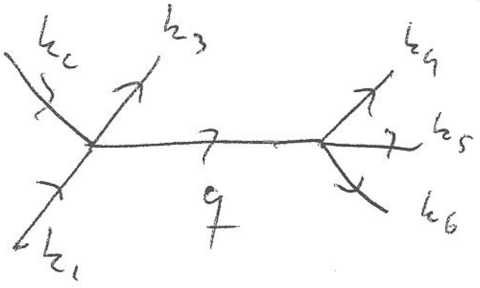
$$(-i\lambda)^4\int \frac{i}{p^2-m^2+i\varepsilon}\,\frac{i}{(k_1+k_2-p)^2-m^2+i\varepsilon}\,\frac{i}{q^2-m^2+i\varepsilon}\,\frac{i}{(p-q-r)^2-m^2+i\varepsilon}\,\frac{i}{r^2-m^2+i\varepsilon}\,\frac{i}{(k_1+k_2-r)^2\cdots}\,\frac{dq\,dp\,dr}{(2\pi)^4}$$

Now let's explore some consequences of our formulation in terms of diagrams. Let's go back to the diagram, with amplitude

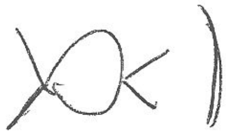


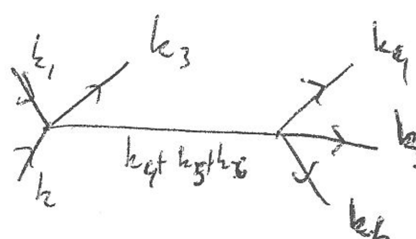
$$= \frac{1}{2}(-i\lambda)^2 \int \frac{i}{k^2-m^2+i\varepsilon} \frac{i}{(k_1+k_2-k)^2+i\varepsilon} \frac{dk}{(2\pi)^4}$$

For large k this integral goes as $\int \frac{dk}{k^4}$ and diverges. We'll learn how to fix this problem later when we deal with renormalization. For now we just ignore it, but before moving one let's notice that



$$= (-i\lambda)^2 \frac{i}{(k_4+k_5+k_6)^2-m^2+i\varepsilon}$$ doesn't

divergence since it has no variable which is integrated over. Recalling that variables which are integrated are those not fixed by momentum conservation on each vertex, we see that it is useful to distinghish between two classes of graphs/diagrams tree graphs (like ) and loop graphs/diagrams (like ), the former are those containing no closed loops, the later are those containing closed loops. Notice that graphs which contain integration over momenta are precisely the loop diagrams. Therefore the problem of divergences arises on loop diagrams only.

Consider again 

$$= (-i\lambda)^2 \frac{i}{(k_4+k_5+k_6)^2 - m^2 + i\varepsilon}$$

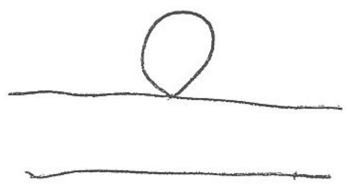
The physical process is clear: two mesons with momenta k_1 and k_2 interact and ~~give~~ ~~birth~~ + give origin to two other mesons with momenta k_3 and $q = k_4+k_5+k_6$. Now notice that although $k_i^2 = m^2$ for $i=1,..,6$, there is no reason why $q^2 = m^2$. Recall that particles for which $k^2 = m^2$ is satisfied are called on-shell and particles for which $k^2 \neq m^2$ are said to be off-shell.

Off-shell particles are also called <u>virtual particles.</u> Now the important thing to notice is that the farther q^2 is from m^2 the smaller the amplitude. So, real particles (i.e, for which the on-shell condition holds) contribute more to the amplitude (and consequently to the scattering) than ~~at~~ virtual ones; and the more "virtual" it is the lesser its contribution. Notice that the same argument holds for loop diagrams: $\frac{1}{2}(-i\lambda)^2 \int \frac{i}{k^2 - m^2 + i\varepsilon} \frac{i}{(k_1+k_2-k)^2 + i\varepsilon} \frac{dk}{(2\pi)^4}$

Again, the integrand is large only if one or the other (or both) of virtual particles associated with the two internal lines are closed to be real.

Now let me turn to a claim made before: that we only ~~had~~ need to worry about connected diagrams.

The idea is that disconnected diagrams don't really contribute to the scattering. For example:



corresponds to a process where the two mesons don't interact; there is the creation an annihilation of a virtual particle at some point, but no interaction between the incoming and outgoing particles.