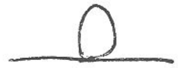
7. Renormalization

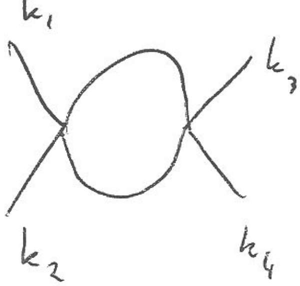
Diagrams with closed loops are divergent quantities due to integration over internal momentum. For example



$$\propto \lambda \int \frac{1}{k^2 - m^2} \frac{dk}{(2\pi)^4}$$

where we omit the $i\varepsilon$ term (Notice that this corresponds to the "free propagator" $\Delta(0)$ at first order, i.e, it contributes to the diagonal part of the S-matrix. Recall that we don't write some terms such as the overall conservation δ-fc etc). There are four powers of k in the numerator (due to dk), two in the numerator, so the integral diverges quadratically at large k (high momentum = high energy = ultraviolet divergence) — we won't bother now with divergences which occur at $k \to 0$, infrared divergences.)

Another example:



$$\propto \lambda^2 \int \frac{1}{(k^2-m^2)(k_1+k_2-k)^2} \frac{dk}{(2\pi)^4}$$

We have four powers of k on both numerators and denominator so it diverges logarithmically.

We see that it may be useful ~~to~~ to define a "degree of divergence", which tells us how the integral diverges — or more precisely, how it behaves for large k, quadratically etc.

Consider a diagram of order n, i.e., n-vertices, E external lines, I internal lines, L loops, and consider space-time with dimension d. Then each propagator associated with an internal line contributes with a power of 2 in the denominator and each loop ~~to~~ with a power of d in the numerator (recall that the number of loops equals the number of momenta not fixed at each vertex and these are integrated over). We define the superficial degree of divergence:

$$D = dL - 2I.$$

This gives $D = 2$ and $D = 0$ in the above examples, as expected. The name superficial is because this will give necessary but not sufficient conditions for convergence of integrals. Now let's express D in terms of E and n.

There are I internal momenta; there is momentum conservation at each vertex (of which there are n) plus an overall momentum conservation so there are $n-1$ relations between the momenta. So the number of independent momenta is $I - (n-1)$; but this is also L so

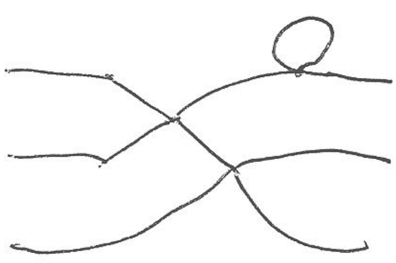
$$L = I - n + 1$$

In φ^4-theory each vertex gives four legs so there are $4n$ legs, some external and some internal, but the internal ones count twice, so

$$4n = E + 2I$$

Solving for E and n gives $D = d(I - n + 1) - 2\left(\frac{4n - E}{2}\right)$ or $D = d - \left(\frac{d}{2} - 1\right)E + n(d-4)$

In the case $d = 4$ this gives $D = 4 - E$ which is independent of n (again we get the correct result for  and ). This seems to suggest that any graph with $E > 4$ would convergege, but that clearly is not the case since we know that graphs with internal loop may diverge, e.g.



To better understand what kind of information D gives, let's go back to the general expression for D:

$$D = d - \left(\frac{d}{2} - 1\right) E + n(d-4)$$

Recall that $n = \#$ of vertices = order in perturbation theory. So this expression tells us that if the coefficient of n is > 0 the divergences become ~~worst~~ worse with increasing n, i.e., the more we add terms in the perturbation expansion the worse the divergence is. In $d=5$, f.ex, we see that the theory contains an ~~infinit~~ infinite number of terms, each one more divergent than the one before (order one it diverges linearly, order two quadratically and so one), so it seems to be hopeless to try to fix the theory order by order. In $d=4$, however, D does not depend on n, so even though some graphs diverge there is hope that we can fix the theory order by order.

That's why D is called "superficial", i.e., it gives necessery but not sufficient conditions to "control" the divergencies — when such control is possible (and we'll see how) the theory is called renormalizable.

Notice that in $d=3$ we have

$$D = 3 - \frac{1}{2}E - n$$

, so D decreases with n. Hence for a given E there are only finitely many divergent graphs — such theories are called super-renormalizable.

It's worth to write D for a φ^r theory in d-dimensions:

$$D = d - \left(\frac{d}{2} - 1\right)E + n\left(\frac{r}{2}(d-2) - d\right)$$

In $d=4$:

$$D = 4 - E + n(r-4)$$

We see that φ^6 theory is non-renormalizable and φ^3 is super-renormalizable. It's curious to see that in $d=2$ we have

$$D = 2 - 2n$$

i.e, independent of r.

It's worth to mention that although D doesn't give sufficient conditions for renormalizability, there is something called Weinberg's thm which says that if the degree of divergence D together with the degree of divergence of all its subgraphs is negative then the Feynman diagram converges.

Before introducing the renormalization schemes it's useful to have some more terminology and shortcuts which help us to decide whether a theory is renormalizable.

We start with dimensional analysis. In units such that $c = \hbar = 1$, the action in d-dimensions is dimensionless $S = \int \mathcal{L}\, dx$, so $[\mathcal{L}] = L^{-d}$ (L = length). Since in this units mass and momentum have units of inverse of length we can also write $[\mathcal{L}] = \Lambda^d$ (Λ = momentum) (See St p. 6).

We refer to Λ as "mass dimension". Since $[\mathcal{L}] = [\partial^\mu \varphi \partial_\mu \varphi]$ and $[\partial_\mu] = L^{-1}$ we have $[\varphi] = L^{1-d/2}$ or $[\varphi] = \Lambda^{\frac{d}{2}-1}$. Consider now the term $\lambda \varphi^r$. If $[\lambda] = L^{-\delta} (= \Lambda^\delta)$, since $[\lambda \varphi^r] = [\mathcal{L}] = L^{-d} (= \Lambda^d)$ we get:

$$-\delta + r\left(1 - \frac{d}{2}\right) = -d \qquad \text{so} \quad \delta = d + r - \frac{rd}{2}$$

For example, the coupling λ has the following dimensions in following theories:

$\lambda\varphi^4$: $\delta = 4 - d$, $[\lambda] = \Lambda^{4-d}$

$\lambda\varphi^3$: $\delta = 3 - \frac{d}{2}$, $[\lambda] = \Lambda^{3-\frac{d}{2}}$

$\lambda\varphi^6$: $6 - 2d$, $[\lambda] = \Lambda^{6-2d}$

Using $\delta = d + r - r\frac{d}{2}$ to solve for r and plugging in $D = d - \left(\frac{d}{2} - 1\right)E + n\left(\frac{r}{2}(d-2) - d\right)$ (see p. 88) we get

$$D = d - \left(\frac{d}{2} - 1\right)E - n\delta$$

Hence a necessary condition for renormalizability is that the coupling constant have mass dimension $\delta \geq 0$ (e.g., Fermi theory of weak interactions has $\delta = -2$ and hence is non-renormalizable Ry 312).

As an application, consider the following situation. Suppose we have a QFT with fields $\varphi_1, \ldots, \varphi_N$ and $\mathcal{L} = \int\left(\sum_i Q_i(\varphi_i) + \sum_k \lambda_k I_k(\varphi_1 \ldots \varphi_N)\right)$ where Q_i is the free part and I_k are the interaction terms. Given a Feynmann diagram of the theory, denote by v_k the number of internal vertices of type I_k, i.e, internal vertices corresponding to the interaction I_k; let e_i denote the number of external vertices corresponding to φ_i, then ([Wi], 436)

$$D = d - \sum_i e_i[\varphi_i] - \sum_k v_k[\lambda_k]$$

Now suppose gravity in an open set of $\mathbb{R}^d$, the Lagrangian is $\mathcal{L}(g) = \frac{1}{16\pi G}\int R_g\, dx$ where G is Newton's constant. If we write $g_{ij} = \delta_{ij} + \sqrt{G}\, h_{ij}$, the Lagrangian density is $\mathcal{L} = \frac{R_g}{16\pi G}$

So in terms of h:

$$\mathcal{L} = \frac{1}{16\pi}\left((\nabla h)^2 + \sqrt{G}\, h\, (\nabla h)^2 + \dots \right)$$

Now $[\mathcal{L}] = \Lambda^d$, so $[\nabla h]^2 = \Lambda^d$. Since $[\nabla] = \Lambda$, we get $\Lambda^2 [h]^2 = \Lambda^d$, so $[h] = \Lambda^{\frac{d-2}{2}}$. It follows:

$$[\sqrt{G}][h][\nabla h]^2 = \Lambda^d$$

$$\sqrt{[G]}\, \Lambda^{\frac{d-2}{2}}\, \Lambda^2 \left(\Lambda^{\frac{d-2}{2}}\right)^2 = \Lambda^d$$

$$\sqrt{[G]}\, \Lambda^{\frac{d-2}{2}} = 1 \Rightarrow [G]^{1/2} = \Lambda^{\frac{-d+2}{2}}$$

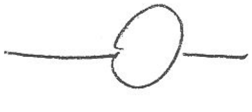
so $[G] = \Lambda^{-d+2}$. From the above formula we see that gravity is non-renormalizable if $d > 2$

Another useful concept is that of "(one-particle) irreducible" diagrams. Consider the following diagram in φ^3 theory:

Since each line is associated with a propagator $\frac{i}{k^2-m^2+i\epsilon}$, it seems intuitive that we should be able to determine this diagram from the more elementary parts  and , since they are joined by a single line which corresponds to a propagator of the form $\frac{i}{k^2-m^2+i\epsilon}$. As we'll see it turns out that this is indeed the case what motivates the definition of two subclasses of graphs. We call a "truncated" diagram one which the propagators associated with the external lines have been removed. Remark: recall that we add an extra Feynman rule which was to ommit propagators associated with external lines, but this was a shorthand way of writting the associated green's fc; the external propagators were implicitly understood. Here, on truncated diagrams there are no external propagators at ~~all~~ all.

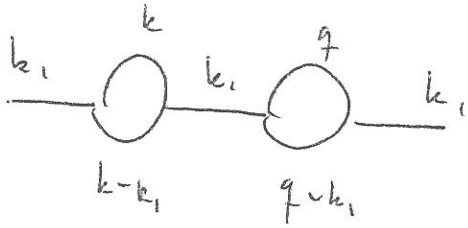
There is no universal notation for truncated diagrams, we'll follow ST and write them with bars on external legs, e.g.:

 is a truncated diagram for

 .

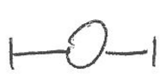
"One-particle irreducible" diagrams (1PI) are those truncated diagrams which cannot be disconnected by cutting and edge (recall that we don't consider disconnected diagrams, so whenever we said "diagram" it should be understood "connected diagram"), eg,  is 1PI but  is not.

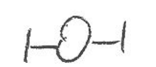
As a very simple example, consider



which occurs in order λ^4 in φ^3 theory. Ommiting symmetry factors and powers of ~~m~~ i the Green's function is:

$$\lambda^4 \frac{1}{k_1^2-m^2+i\varepsilon}\frac{1}{k_1^2-m^2+i\varepsilon}\frac{1}{k_1^2-m^2+i\varepsilon}\int\frac{1}{k^2-m^2+i\varepsilon}\frac{1}{(k-k_1)^2-m^2+i\varepsilon}\frac{dk}{(2\pi)^4}\int\frac{1}{q^2-m^2+i\varepsilon}\frac{1}{(q-k_1)^2-m^2+i\varepsilon}\frac{dq}{(2\pi)^4}$$

$= \lambda^4 \dfrac{1}{k_1^2-m^2+i\varepsilon}$  $\dfrac{1}{k_1^2-m^2+i\varepsilon}$  $\dfrac{1}{k_1^2-m^2+i\varepsilon}$

So the knowledge of  is enough to determine the Green's function. We'll show that not only this is always the case but also that expressing everything in terms of 1PI diagrams leads to a perturbative expansion with more physical appeal and easier to organize according to the renormalization schemes we want to implement.

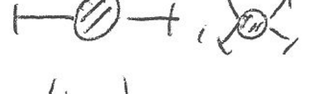
Define the n-point vertex function $\Gamma(k_1, \ldots, k_n) \equiv \Gamma^{(n)}(k)$ as (Ry 258, St 91; St uses a different sign convention, but here and in what follows we follow Ry convention)

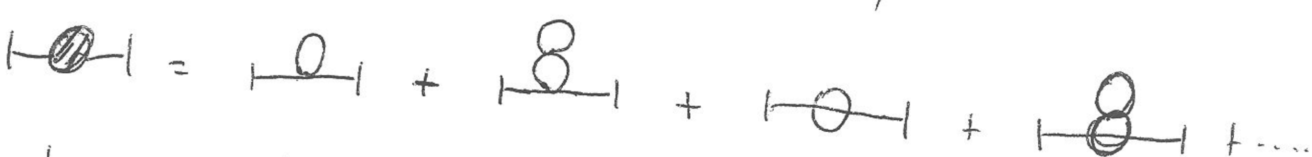
$$\Gamma(k_1, \ldots, k_n)\, G_c(k_1, \ldots, k_n) = i$$

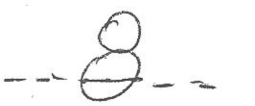
where the subscript c on the n-point function G is to remind us that we're dealing only with connected Green's function.

At first glance Γ seems to be of no interest since it's simply i/G_c, but it turns out that Γ is expressible entirely in terms of 1PI diagrams. Hence it's easier to compute and renormalize Γ instead of G_c, and once that's done we can get the renormalized G_c by i/Γ.

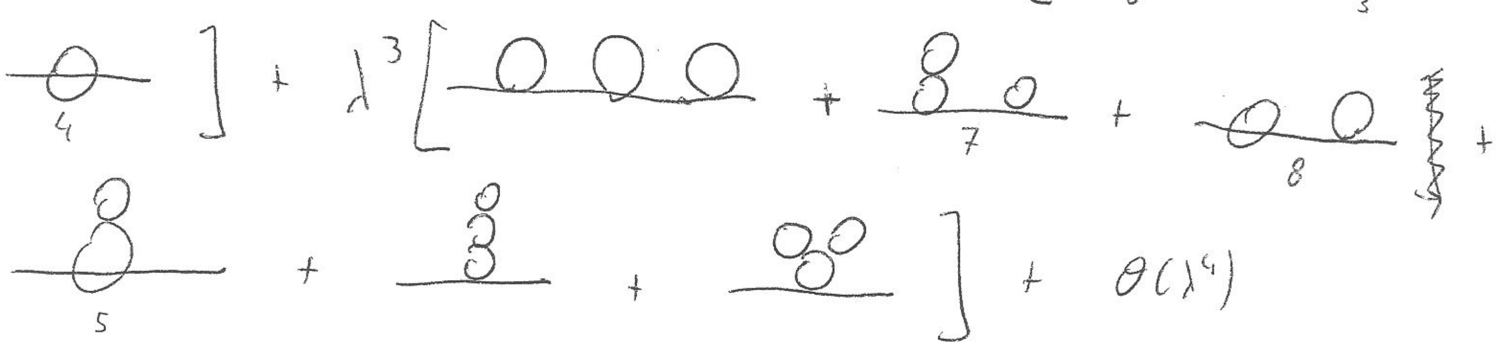
Let's go back to φ^4 theory.

Let $\frac{1}{i}\Gamma_n$ denote the sum of one particle irreducible diagrams with n truncated external lines, graphically denoted  (remark: there is no universal convention here either), so for example

$-i\Gamma_2 =$ 

(Sometimes truncated diagrams are denoted with the removed external legs by dots, eg, $-i\Gamma_2 =$   $+ \dots$). On the other hand the two point function (the connected one!) is, to all orders:

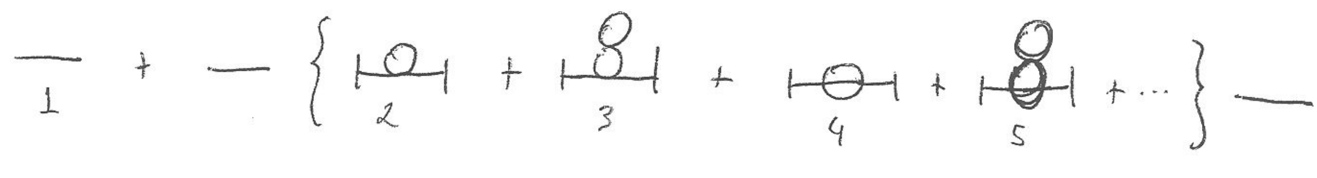
$G_c^{(2)} = G_c^{(2)}(k_1, k_2) =$ 

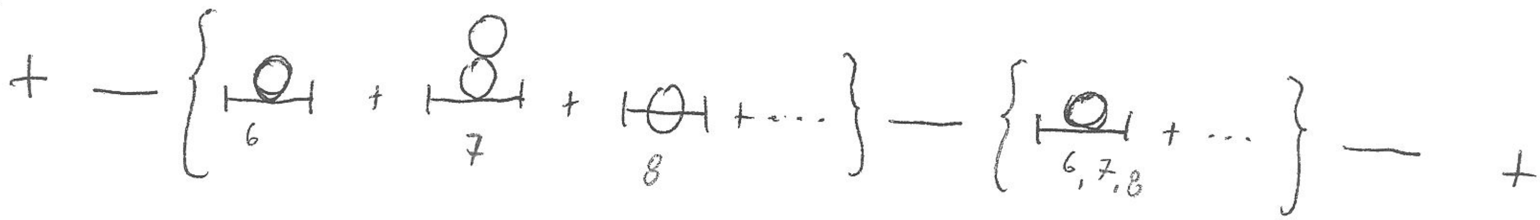


$=$ 

caution: do not confuse  with 

But we can rewrite (the coupling etc is included on the 1PI diagrams):

$G_c^{(2)} =$ 

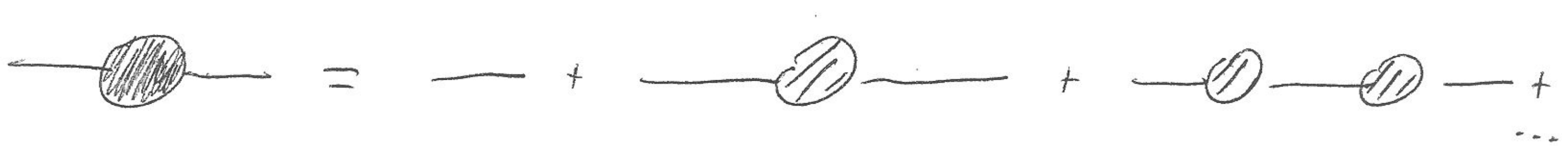


...

Now each of the sums inside $\{\ \}$ is $\frac{1}{i}\Gamma_2$, so, since $\frac{1}{i}\Gamma_2 =$  we have

$G_c^{(2)} =$ 

or



and hence we are expressing $G_c^{(2)}$ in terms of 1PI and connecting edges (which corresponds to propagators $\frac{i}{k^2-m^2+i\varepsilon}$). Since —— $= \frac{i}{k^2-m^2}$

(more precisely $\frac{i}{k^2-m^2+i\varepsilon}$, but we drop $i\varepsilon$ from now on ~~&~~ since the integral diverges anyway), whe we have, abreviating $G_0 = \frac{i}{k^2-m^2}$:

$$G_c^{(2)} = G_0 + G_0 \frac{\Gamma_2}{i} G_0 + G_0 \frac{\Gamma_2}{i} G_0 \frac{\Gamma_2}{i} G_0 + \dots$$

$$= G_0 \left(1 + \frac{\Gamma_2}{i} G_0 + \frac{\Gamma_2}{i} G_0 \frac{\Gamma_2}{i} G_0 + \dots \right)$$

$$= G_0 \frac{1}{1 - \frac{\Gamma_2}{i} G_0} = \cancel{G_0} \frac{1}{\cancel{G_0}} \frac{1}{\frac{1}{G_0} - \frac{\Gamma_2}{i}} = \frac{i}{\frac{i}{G_0} - \Gamma_2}$$

$$= \frac{i}{k^2 - m^2 - \Gamma_2}$$

, and so we have a simple expression for the two point function is terms of Γ_2

$$G_c^{(2)} = \frac{i}{k^2-m^2-\Gamma_2}$$

It follows that the 2-point vertex function is $\Gamma^{(2)} = k^2 - m^2 - \Gamma_2$.

For the four-point function the combinatorics is, of course, more complicated, but proceeding analogously to what we did we get

$$G_c^{(4)} = \frac{i}{-\Gamma_4}, \quad \text{from which } \Gamma^{(4)} = -\Gamma_4$$

Indeed we get $G_c^{(n)} = \dfrac{i}{-\Gamma_n}$, $\Gamma^{(n)} = -\Gamma_n$, $n \geq 3$, n even (recall $G_c^{(n)} = 0$ for n odd).

The difference from the case $n=2$ to higher order ones, is that for $n=2$ the term in order λ^0 is connected (being —) so it contributes to $G_c^{(2)}$, while for $n>2$ (e.g., $n=4$ we have $=$) is disconnected, hence it does not contribute to $G_c^{(n)}$ and that's why the term k^2-m^2 does not appear.

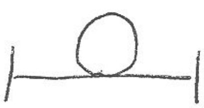
We are in a ~~cond~~ position now where we can start the process of renormalization. We first apply a procedure called dimension regularization. (regularization means a method of isolating the divergences for a subsequent renormalization, i.e., the removal of the divergences.) The idea is to treat integrals on d-dimensions and then take the limit $d \to 4$. It turns out that singularities of 1-loop graphs are simple poles in $d-4$.

In d-dimensions φ has mass dimension $\frac{1}{2}d - 1$ and L has mass dimension d, so $[\lambda] = \Lambda^{4-d}$ (see 91) and hence it is dimensionless in $d=4$, but to keep it dimensionless in d dimensions we introduce an arbitrary mass paramer μ:

$$\mathcal{L} = \frac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\mu^{4-d}\lambda}{4!}\varphi^4$$

(for the formulas which will be used below in computing the integrals, see Ry 314ff)

We start with the divergent term:

 $$= \frac{1}{2}\lambda\mu^{4-d}\int \frac{1}{k^2-m^2}\frac{dk}{(2\pi)^d}$$ where the integral is over d-dimensional momentum space ($\frac{1}{2}$ is the symetry factor). This term contributes in first order to the 2-point function. This integral can be computed to produce

 $$= -\frac{i\lambda}{32\pi^2}\, m^2 \left(\frac{4\pi\mu^2}{-m^2}\right)^{2-\frac{d}{2}} \Gamma\left(1-\frac{d}{2}\right)$$

where Γ is the gamma function (do not confuse with the vertex function $\Gamma^{(2)}$). The gamma function has poles at zero and negative integers, so we see that the divergence of the integral manifests itself as a simple pole as $d \to 4$. Using

$$\cancel{\Gamma(-n+\varepsilon) = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \gamma}$$

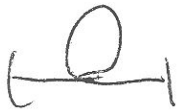
$$\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\varepsilon} + \psi_1(n+1) + O(\varepsilon)\right]$$

where $\psi_1(n+1) = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \gamma$, with $\gamma =$
$= -\psi_1(1) = 0.577$ (Euler-Mascheroni constant).

Putting $\varepsilon = 4-d$, gives

$$\Gamma\left(1-\frac{d}{2}\right) = \Gamma\left(-1+\frac{\varepsilon}{2}\right) = \frac{(-1)^1}{1!}\left[\frac{2}{\varepsilon} + 1 - \gamma + O(\varepsilon)\right]$$

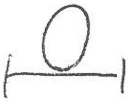
So



$$= \frac{-i\lambda}{32\pi^2} m^2 \left(\frac{4\pi\mu^2}{-m^2}\right)^{2-\frac{d}{2}} \Gamma\left(1-\frac{d}{2}\right) =$$

$$= -\frac{i\lambda}{32\pi^2} m^2 \left(\frac{4\pi\mu^2}{-m^2}\right)^{\frac{\varepsilon}{2}} \Gamma\left(-1+\frac{\varepsilon}{2}\right)$$

Using $a^{\varepsilon} = 1 + \varepsilon \ln a + \dots$ and the above expression for $\Gamma\left(-1+\frac{\varepsilon}{2}\right)$:

 $= -\frac{i\lambda}{32\pi^2} m^2 \left[1 + \frac{\varepsilon}{2}\ln\left(\frac{4\pi\mu^2}{-m^2}\right)\right]\left[-\frac{2}{\varepsilon} - 1 + \gamma + \mathcal{O}(\varepsilon)\right]$

$$= \frac{i\lambda m^2}{16\pi^2\varepsilon} + \frac{i\lambda m^2}{32\pi^2}\left[1 - \gamma + \ln\left(\frac{4\pi\mu^2}{-m^2}\right)\right] + \mathcal{O}(\varepsilon)$$

Or  $= \frac{i\lambda m^2}{16\pi^2\varepsilon}$ + finite (notice that the finite part depends on μ).

So we explicitly isolated the lowest order correction divergent contribution to the 2-point function. As we said, we'll work with the vertex 2-pt function instead of the Green function. Recall that

$\Gamma^{(2)} = k^2 - m^2 - \Gamma_2$ and

and that (p. 98).

$$G_c^{(2)} = \frac{i}{k^2 - m^2 - \Gamma_2} = \frac{i}{\Gamma^{(2)}}$$

In the free ~~propagator~~ theory, the mass of the particle corresponds exactly to the pole of the propagator, but we don't get it here (unless $\Gamma_2 = 0$ what is not the case). This suggests that the parameter m above is not the actual mass of the particle; indeed, it is the same m appearing in the free Lagrangian (setting $\lambda = 0$) and hence m would be the mass if no interactions were present; since they always are, m is indeed an unobservable quantity. (Ry 318). Indeed, keeping m as the mass we would neglect contribution to it given by the virtual particles/ self-interaction (find reference) (remark: there is indeed an argument for actual mass = poles, [Wei], 430)

Hence the actual mass should actually be given by the poles of $G_c^{(2)}$, i.e.,

$$\Gamma^{(2)}\Big|_{k^2=m_R^2} = \left[k^2 - m^2 - \Gamma_2\right]\Big|_{k^2=m_R^2} = 0$$

Or $$m_R^2 + \Gamma_2\Big|_{k^2=m_R^2} = m^2 .$$

So, we want to compute $\Gamma^{(2)}$, what means to compute Γ_2, but from the above equation we see that we need m_R, but m_R is given by the pole, or $\Gamma \equiv 0$, so to find it we need Γ_2. It turns out that we can escape this "loop" by computing the desired quantities, order by order in a sort of iteration..

We had $-i\Gamma_2 =$  $+$  $+ \ldots$

So to first order in λ

$$\Gamma^{(2)} = k^2 - m^2 + \frac{1}{i}$$ 

$$= k^2 - m^2 + \frac{\lambda m^2}{16\pi^2 \varepsilon} + \text{finite}\,,$$ so for

$m_1^2 = k^2$ (we use m_1 instead of m_R cause m_1 is the renormalized mass up to first order)

$$0 = m_1^2 - m^2 + \frac{\lambda m^2}{16\pi^2 \varepsilon}$$ and hence, to first order

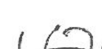
$\Gamma^{(2)} = k^2 - m_1^2$. Notice that, to first order

$$m^2 = m_1^2 + \frac{\lambda m^2}{16\pi^2\varepsilon} = m_1^2 + \frac{\lambda}{16\pi^2\varepsilon}\left(m_1^2 + \frac{\lambda m^2}{16\pi^2\varepsilon}\right)$$

$$= m_1^2\left(1 + \frac{\lambda}{16\pi^2\varepsilon}\right)$$ and we see that indeed $m = \infty$ when $\varepsilon \to 0$ if m_1 is to remain finite.

So, we have $\Gamma^{(2)} = k^2 - m_1^2$ is the 2-point vertex function to first order. If ~~first~~ we stop at first order, then this is the answer and m_1 is the physical mass, otherwise ~~the~~ introducing higher order terms requires a redefinition of m_1 (which now becomes infinite in the limit $\varepsilon \to 0$), exactly as ~~to~~ m was redefined before. However, the next second order contribution will be given by

 +  (notice the first term consists essentially of a product of ~~the~~ integrals like  ~~across~~ it ~~should give~~ ~~a second contribution~~, i.e.,

$\Gamma^{(2)} = k^2 - m^2 + \frac{1}{i}\Big($  $+$  $+$  $\Big)$

$= k^2 - m_1^2 + \frac{1}{i}\Big($  $+$ $\Big)$

It follows that only renormalizing the mass is not enough to render $\Gamma^{(2)}$ finite, and renormalizing λ and introducing a field renormalized constant is also necessary.

<u>Remark:</u> notice that the finite part depends on μ, it is also necessary to show that physical quantities do not depend too on μ (RG flow, Ry 319)

All this is better understood by using another renormalization scheme, that of counter-terms.

We first need to reorganize our expansion. We'll apply renormalization not only order by order in perturbation theory but also loop by loop, i.e., remove divergences on n-pt functions at 1-loop, then 2-loops etc.

This will not only introduce a more systematic approach but also turn the physics clearer: an expansion in loops can also be regarder as an expansion in powers of $\hbar$. To see this, we go ~~back~~ to standard units and recall:

$$Z[J] = e^{\frac{1}{\hbar}\mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J}\right)} Z_0[J]$$

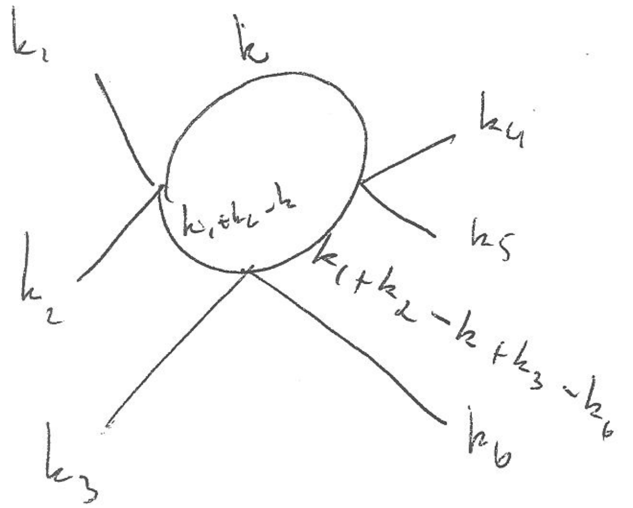
where $Z_0[J] = N e^{-\frac{1}{2} i \hbar \int dx\, dy\, J(x)\, \Delta(x-y)\, J(y)}$

From these two expressions we see that each vertex contributes a factor $\hbar^{-1}$ (indeed $\lambda\varphi^4 \mapsto \frac{1}{\hbar}\mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J}\right) = \frac{1}{\hbar}\left(\frac{\lambda}{i}\frac{\delta}{\delta J}\right)^4$, so $e^{\frac{1}{\hbar}(\;)} = 1 + \frac{1}{\hbar}\left(\frac{\lambda}{i}\frac{\delta}{\delta J}\right) + \frac{1}{2}\left[\frac{1}{\hbar}\left(\frac{\lambda}{i}\frac{\delta}{\delta J}\right)\right]^2 + \dots$ and λ^n appears in n^{th} order) and each propagator contributes a factor $\hbar$, to a general graph in n^{th} order.

hence a graph with n vertex and I internal lines has a factor $\hbar^{I-n}$, ~~but~~ $I-n = L-1$ so $\hbar^{L-1}$.

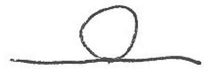
Now, among the n-point vertex functions $\Gamma^{(n)}$, for each fixed number of loops we need to ~~~~ worry ~~~~ ~~deal~~ about $\Gamma^{(2)}$ and $\Gamma^{(4)}$ only. The reason is the following Ra, 143, Zee 161). Consider any loop residing inside a diagram, integration over the loop gives rise to divergencies if the loop is bounded by one or two propagators (internal lines) at most.

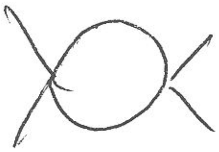
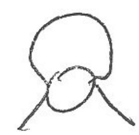
For example, the following diagram has one loop made out of 3 internal lines



$$\propto \int \frac{1}{k^2-m^2}\,\frac{1}{(k_1+k_2-k)^2-m^2}\,\frac{1}{(k_1+k_2-k+k_3-k_6)^2-m^2}\,\frac{d^4k}{(2\pi)^4} \sim \int \frac{d^4k}{k^6}$$

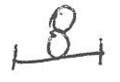
which is free of ultraviolet divergences (we don't look at IR divergenges here)

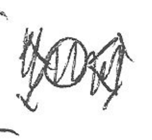
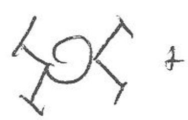
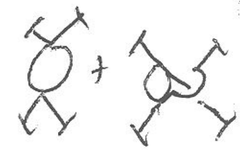
A loop bounded by one propagator involves one vertex only:  leaving two free legs, which may in turn be attached to the rest of the diagram (or one external, one attached). In this case we can isolate this two pt function from the insides of the diagram. A loop bounded by two propagators involves two vertices and therefore four legs.

 which can be attached to the rest of the diagram — and one is led to ~~isola~~ isolate from the diagram a four point function (or a two point function if two of the four legs are attached together, i.e., ). In this way we see that UV divergences inside a diagram originate from such loops and such loops appear in two- and four-point functions nested inside the diagram. This means that the generic sources of divergences are only 2- and 4-point functions

(define primitive divergences)

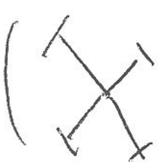
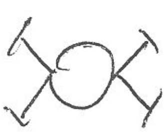
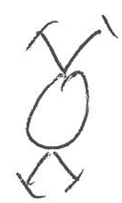
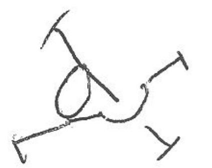
Now we can implement the renormalization procedure. We recall:

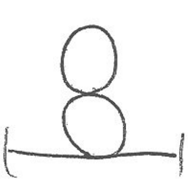
$$\Gamma^{(2)} = k^2 - m^2 + \frac{1}{i}\Big($$  $+$  $+$  $+ \cdots \Big)$

$$\Gamma^{(4)} = +\frac{1}{i}\Big[$$  $+$  $+$  $+$  $+$  $+ \cdots \Big]$

 $$= \frac{i\lambda m^2}{16\pi^2 \varepsilon} + \frac{i\lambda m^2}{32\pi^2}\left[\psi_1(2) + \ln\left(\frac{4\pi\mu^2}{-m^2}\right)\right]$$

and, as a calculation shows (see Ry 315ff, Ra, 169)

$\Big($  $+$  $+$  $+$  $\Big)$ $$= -i\lambda\left(1 - \frac{3\lambda}{16\pi^2\varepsilon}\right)\mu^{\varepsilon} + \text{finite}$$

 $$= -\, i m^2 \left(\frac{\lambda}{16\pi^2}\right)^2\left(\frac{1}{\varepsilon^2} + \frac{1}{2\varepsilon}\left(\psi_1(2) + \psi(1)\right) + 2\ln\left(\frac{4\pi\mu^2}{-m^2}\right) + \cdots\right)$$

Let's start with one loop. We have

$$\Gamma^{(2)} = k^2 - m^2 + \frac{1}{i}$$

$$= k^2 - m^2 + \frac{\lambda m^2}{16\pi^2 \varepsilon} + \frac{\lambda m^2}{32\pi^2}\left[\psi(2) - \ln\left(\frac{-m^2}{4\pi\mu^2}\right)\right]$$

Which diverges as $\varepsilon \to 0$.

Notice that $k^2 - m^2$ comes from the free part of the Lagrangian (indeed $\Gamma^{(2)} = k^2 - m^2$ for the free theory). So, if we change the term m^2 to $m^2 + \delta m^2$ in the Lagrangian, we get $m^2 + \delta m^2$ instead of m^2 in $\Gamma^{(2)}$. If we choose $\delta m^2 = \frac{\lambda m^2}{16\pi^2 \varepsilon}$ we then get $\Gamma^{(2)} = k^2 - \left(m^2 + \frac{\lambda m^2}{16\pi \varepsilon}\right) + \frac{\lambda m^2}{16\pi \varepsilon} + \frac{\lambda m^2}{32\pi^2}[\ldots]$ $= k^2 - m^2 + \text{finite}$. Notice that the change $m^2 \mapsto m^2 + \delta m^2$ would affect all terms containing m in $\Gamma^{(2)}$, but the other ones would change by $\mathcal{O}(\lambda^2)$ so we keep them as they were.

So by changing m^2 by $m^2 \mapsto m^2 + \delta m^2$, or, equivalently, by adding to the Lagrangian a term $\delta \mathcal{L}_1 = -\frac{\lambda m^2}{2 \cdot 16\pi^2 \varepsilon} \varphi^2 = -\frac{\delta m^2}{2} \varphi^2$ we cancel the $O(\lambda)$ divergence in $\Gamma^{(2)}$. Notice, and this will be the key idea for the correction order by order, that ~~[illegible]~~ we modify the term in order $O(\lambda^0)$ to cancel a divergence in $O(\lambda)$; this modification doesn't change $O(\lambda)$ terms because if we did so they would become $O(\lambda^2)$ ~~the~~ terms – which we don't consider at this level of approximation.

The next relevant ~~and~~ one-loop divergent diagram at $O(\lambda)$ is ~~the~~ $\Gamma^{(4)}$; Notice that the first loop in $\Gamma^{(4)}$ happens at λ^2.

$$\Gamma^{(4)} = -i\lambda\mu^{\varepsilon}\left(1 - \frac{3\lambda}{16\pi^2\varepsilon}\right) + \text{finite}$$

Now we do something similar to what we did with with m^2: change $\lambda \mapsto \lambda + \frac{3\lambda^2\mu^{\varepsilon}}{16\pi^2\varepsilon}$, or, equivalently, add $\delta\mathcal{L}_\lambda = -\frac{1}{4!}\frac{3\lambda^2\mu^{\varepsilon}}{16\pi^2}\varphi^4$ to the Lagrangian. To keep the analogy of what we did with m^2: before, to remove a divergence of $\mathcal{O}(\lambda)$ in Γ we modify the term $\mathcal{O}(\lambda^0)$ of Γ by adding a term of $\mathcal{O}(\lambda)$ which cancels the divergence. Now to remove a divergence of $\mathcal{O}(\lambda^2)$ we'll modify the $\mathcal{O}(\lambda)$ term of Γ by adding a term of $\mathcal{O}(\lambda^2)$ (as before, we actually modify all orders of Γ, but only the $\mathcal{O}(\lambda)$ will be modify in the order of approximation we are considering)

So $\Gamma^{(4)}$ becomes

$$\Gamma^{(4)} = -i\mu^{\varepsilon}\left(\lambda + \frac{3\lambda^2\mu^{\varepsilon}}{16\pi^2\varepsilon}\right)\left(1 - \frac{3}{16\pi^2\varepsilon}\left(\lambda + \frac{3\lambda^2\mu^{\varepsilon}}{16\pi^2\varepsilon}\right)\right)$$

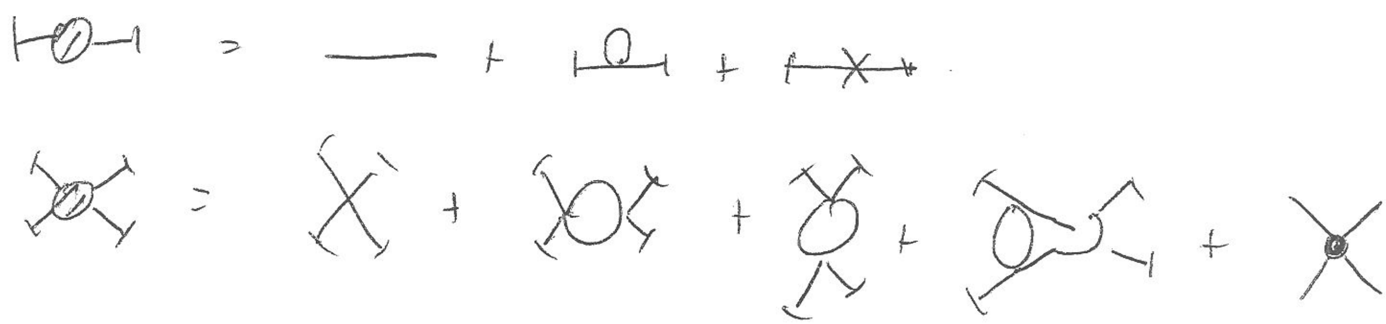
$$= -i\mu^{\varepsilon}\left(\lambda - \frac{3\lambda^2}{16\pi^2\varepsilon} + \frac{3\lambda^2\mu^{\varepsilon}}{16\pi^2\varepsilon}\right) + O(\lambda^3)$$

$$= -i\mu^{\varepsilon}\lambda$$

As always, it's customary to write things in terms of diagrams, so we want to introduce new diagrams which represent the counterterms in Γ. Since the free propagator is and $\frac{\lambda m^2}{16\pi\varepsilon}$ is the modification to the free part ~~that~~ k^2-m^2 we denote  $= \frac{\lambda m^2}{16\pi\varepsilon}$, where $\times$ is to remind us that this term contains a λ (recall that we have a λ for each vertex).

We also denote ~~$3\lambda^2\mu$~~  $= \frac{3\lambda^2\mu^{\varepsilon}}{16\pi^2\varepsilon}$

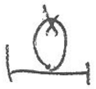
~~so~~ where • is to remind us that this term contains a λ^2 and so is "like" a loop (it's not an actual loop cause it contains no integration). So, in diagrams, ~~to~~ up to one loop:



With this notation it is to be understood that × ~~was~~ comes from the modification of m^2 and • from the modification on λ.

Now let's investigate what happens in 2 loops. Again, we start by $\mathcal{L}^{(2)}$. ~~...~~

~~...~~: At exactly 2 loops the smaller order is λ^2

but of course, now that we modify $\mathcal{L}$ ,  will no longer be the same (as we saw before, but we disregard this modification cause it happen on $\mathcal{O}(\lambda^2)$). It's not very difficult to understand how these modifications happen in terms of diagrams.  is going to be modified by the change we did in the mass (diagramatically $\times$) and the change we did on λ (diagramatically $\bullet$), so this two modifications (equivalently, these two new terms in the Lagrangian) give  and , where  is essentially  but with the "new" mass δm^2 in place of m^2.

Let us see how this works. We had (p. 113)

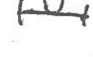
$\frac{1}{i}$  $= \dfrac{\lambda m^2}{16\pi^2\varepsilon} + \dfrac{\lambda m^2}{32\pi^2}\left[\psi_1(2) + \ln\left(\dfrac{4\pi\mu^2}{-m^2}\right)\right]$

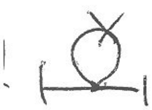
Since  means  with $\delta m^2 = \dfrac{\lambda m^2}{16\pi^2\varepsilon}$ in place of m^2, we get

$\frac{1}{i}$  $= \left(\dfrac{\lambda}{16\pi^2}\right)^2 \dfrac{m^2}{\varepsilon^2} + \left(\dfrac{\lambda}{16\pi^2}\right)^2 \dfrac{m^2}{2\varepsilon}\left[\psi_1(2) + \ln\left(\dfrac{4\pi\mu^2}{-m^2}\cdot\dfrac{\varepsilon}{16\pi^2\lambda}\right)\right]$

Using $\ln\left(\dfrac{4\pi\mu^2}{-m^2}\cdot\dfrac{\varepsilon}{16\pi^2\lambda}\right) = -\ln\left(\dfrac{-m^2}{4\pi\mu^2}\;\dfrac{16\pi^2\lambda}{\varepsilon}\right) = -\ln\left(-\dfrac{m^2}{4\pi\mu^2}\right)$

$-\ln\left(\dfrac{16\pi^2\lambda}{\varepsilon}\right) = -\ln\left(-\dfrac{m^2}{4\pi\mu^2}\right) - \ln\left(1 + \left(1 - \dfrac{16\pi^2}{\varepsilon}\lambda\right)\right) =$

$-\ln\left(-\dfrac{m^2}{4\pi\mu^2}\right) - \left(1 - \dfrac{16\pi^2\lambda}{\varepsilon}\right) + O(\lambda^2)$ (using $\ln(1+x) = x + \dots$ with $x = 1 - \dfrac{16\pi^2\lambda}{\varepsilon}$) Since $\dfrac{16\pi^2}{\varepsilon}\lambda$ in this expression will be multiplied by λ^2 when we plug in , This gives rise to a $O(\lambda^3)$ and can be neglected. Noting that $\psi_1(2) = 1 - \gamma = 1 + \psi_1(1)$, so $\psi(2) - 1 = \psi(1)$ we have:

 $$= \left(\frac{\lambda}{16\pi^2}\right)^2 \frac{m^2}{\varepsilon^2} + \left(\frac{\lambda}{16\pi^2}\right)^2 \frac{m^2}{2\varepsilon}\left[\psi_1(1) - \ln\left(\frac{-m^2}{4\pi\mu^2}\right)\right]$$

Now we do the same for , ie, we consider  with λ replaced by $\frac{3\lambda^2}{16\pi^2\varepsilon}$ (notice that we replace λ by $\frac{3\lambda^2}{16\pi^2\varepsilon}$ and not by $\frac{3\lambda^2\mu^\varepsilon}{16\pi^2\varepsilon}$, this is because on $\mathcal{L}$ we have $\frac{1}{4!}\mu^\varepsilon\lambda\varphi^4$ and this gives rise to $\frac{\lambda m^2}{16\pi^2 \varepsilon}$ on  and not to $\frac{\lambda\mu^\varepsilon m^2}{16\pi^2\varepsilon}$. So if we ~~ch~~ add to $\mathcal{L}$ the term $\frac{1}{4!}\mu^\varepsilon \frac{3\lambda^2}{16\pi^2\varepsilon}$ we don't get μ^ε on ). The result is (Ra 169, remark: our ε equals 2ε of Ra)

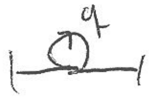
$\frac{1}{i}$  $= 3m^2 \left(\frac{\lambda}{16\pi^2}\right)^2 \left[\frac{1}{\varepsilon^2} + \frac{1}{2\varepsilon}\left(\psi_1(2) - \ln\left(\frac{-m^2}{4\pi\mu^2}\right)\right)\right]$

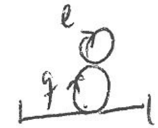
Recall (p. 113)

$\frac{1}{i}$  $= -m^2 \left(\frac{\lambda}{16\pi^2}\right)^2 \left[\frac{1}{\varepsilon^2} + \frac{1}{2\varepsilon}\left(\psi_1(2) + \psi_1(1) - \ln\left(-\frac{m^2}{4\pi\mu^2}\right)\right)\right]$

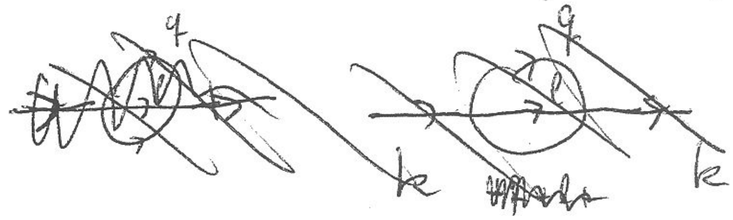
Finally the diagram  can be computed to give (Ra 159 ff):

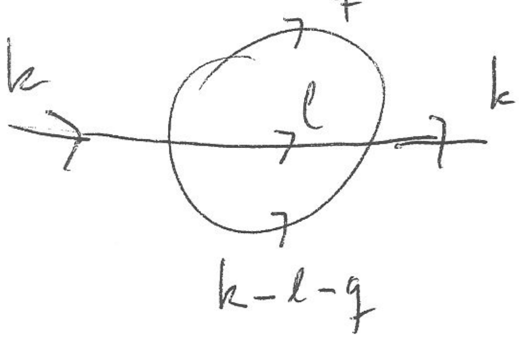
$\frac{1}{i}$  $= -\frac{1}{6}\left(\frac{\lambda}{16\pi^2}\right)^2 \left[\frac{6m^2}{\varepsilon^2} + \frac{6m^2}{\varepsilon}\left(\frac{3}{2} + \psi(1) - \ln\left(\frac{-m^2}{4\pi\mu^2}\right)\right) + \frac{1}{2\varepsilon}k^2 + \dots\right]$

The presence of the term k^2 can be understood without doing the computation of . The terms 

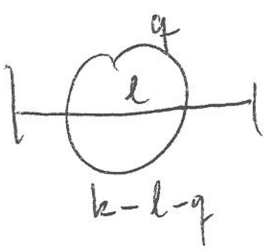
and 

have loops which are integrated over with propagators $\frac{1}{q^2-m^2}$

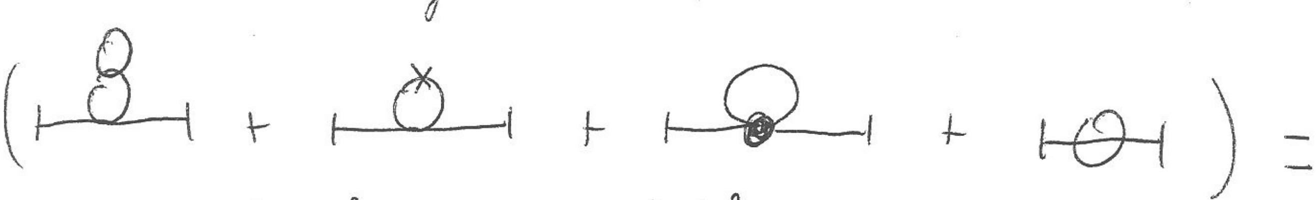
and $\frac{1}{\ell^2 - m^2}$, conservation of momentum in the vertices means an overall delta function which we don't write (as we don't have the propagators of external legs). On: 



, even after cutting the external legs we still have k appearing on the internal loops (and so in the integral):



So, adding all the terms we have, in $\mathcal{O}(\lambda^2)$:



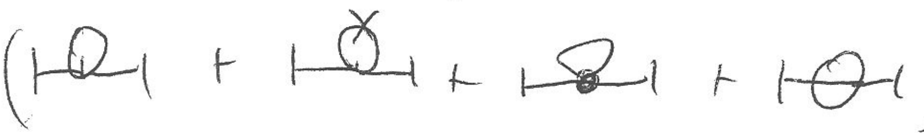
$$= -\frac{1}{12}\left(\frac{\lambda}{16\pi^2}\right)^2 \frac{k^2}{\varepsilon} + \frac{m^2}{2}\left(\frac{\lambda}{16\pi^2}\right)^2 \left[\frac{4}{\varepsilon^2} - \frac{1}{\varepsilon}\right] + \dots$$

A lot of cancelations happened: the $\ln\left(\frac{-m^2}{4\pi\mu^2}\right)$ terms cancel; the ψ_1 can be collected together and using $\psi(n+1) - \psi(n) = \frac{1}{n}$ we get the $-\frac{1}{\varepsilon}$ term. So we have again a divergent expression when $\varepsilon \to 0$.

Again, we add to the Lagrangian a term of the form $\delta\mathcal{L}_3 = -\frac{1}{2}\left[\frac{1}{2}\left(\frac{\lambda}{16\pi^2}\right)^2\left(\frac{4}{\varepsilon^2} - \frac{1}{\varepsilon}\right)\right] m^2\varphi^2$

$= -\frac{1}{2}\,\delta m'^2 \varphi^2$. As we saw this modifies the m^2 in $\Gamma^{(2)}$ by $m^2 \mapsto m^2 + \delta m'^2$ so

$$k^2 - m^2 \mapsto k^2 - m^2 - \left[\frac{1}{2}\left(\frac{\lambda}{16\pi^2}\right)^2\left(\frac{4}{\varepsilon^2} - \frac{1}{\varepsilon}\right)\right] m^2$$

cancelling the divergence on the second term of $\frac{1}{i}\Big($  $\Big)$

For the divergence $-\frac{1}{12}\left(\frac{\lambda}{16\pi^2}\right)^2 \frac{k^2}{\varepsilon}$, notice that ~~with~~ in the free part $\Gamma^{(2)} \mapsto k^2 - m^2 + \ldots$ the term k^2 comes from the ~~kin~~ kinetic term $\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi$ (recall that this term gives rise to $\Box$ which, after Fourier transforming, gives k^2). So if we add to the Lagrangian

$$\delta\mathcal{L}_4 = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi\left[-\frac{1}{12}\left(\frac{\lambda}{16\pi^2}\right)^2\frac{1}{\varepsilon}\right]$$

~~ah~~ we change $k^2 \mapsto k^2 + k^2\left[-\frac{1}{12}\left(\frac{\lambda}{16\pi^2}\right)^2\frac{1}{\varepsilon}\right]$ in $\Gamma^{(2)}$ and this cancels the first divergence above. Hence, we get finite Green's function $\Gamma^{(2)}/G_2$ up to order λ^2 by considering the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi + \frac{1}{2}\left(-\frac{1}{12}\left(\frac{\lambda}{16\pi^2}\right)^2\frac{1}{\varepsilon}\right)\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}\left[m^2 + \overbrace{\frac{\lambda m^2}{16\pi^2\varepsilon}}^{\delta m^2} + \overbrace{\frac{1}{2}\left(\frac{\lambda}{16\pi^2}\right)^2\left(\frac{4}{\varepsilon^2} - \frac{1}{\varepsilon}\right)m^2}^{\delta m'^2}\right]\varphi^2 - \frac{1}{4!}\mu^\varepsilon\left[\lambda + \frac{3\lambda^2}{16\pi^2}\right]\varphi^4$$

So we see that the process of getting finite n-pt functions by adding counterterms is equivalent to start with $\mathcal{L} = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4!}\varphi^4$ and redefining m, λ, and the field φ by

$$\varphi \longmapsto \sqrt{Z_\varphi}\,\varphi \,, \quad Z_\varphi = 1 + \left(-\frac{1}{2}\left(\frac{\lambda}{16\pi^2}\right)^2 \frac{1}{\varepsilon}\right)$$

$$m \longmapsto Z_m\, m \,, \quad Z_m^2 = \frac{m^2 + \delta m^2 + \delta m'^2}{m^2 Z_\varphi}$$

$$\lambda \longmapsto \mu^\varepsilon Z_\lambda \lambda \,, \quad Z_\lambda = \frac{\left(1 + \frac{3\lambda}{16\pi^2}\right)}{Z_\varphi^2}$$

The quantities Z are called <u>renormalization factors</u> and $\varphi_B = \sqrt{Z_\varphi}\,\varphi$, $m_B = Z_m m$, $\lambda_B = \mu^\varepsilon Z_\lambda \lambda$ are the <u>bare field, mass and coupling etc</u>.

The original m^2, φ, λ are the physical quantities (measured values), the bare quantities are infinite quantities with are not measurable.

It seems clear that we can continue this ~~same~~ process ad nauseum and get finite correlation function to arbitrary order $O(\lambda^n)$. The crucial point is that all counter terms $\delta\mathcal{L}_i$ are such that its dependence on φ and its derivatives is the same as that of a term already appearing in $\mathcal{L}$. (Ra 167). So adding these terms means changing (redefining φ, m and λ but we end up with a Lagr. of φ^4 theory. Adding, say, a φ^3 dependence would change $\mathcal{L}$ (and hence the theory), if this is necessary the theory is non-renormalizable.