1. Amplitudes in quantum mechanics

From the path-integral formulation of ordinary quantum mechanics we know that the transition amplitude from position q' at time t' to q'' at time t'', $t'' > t'$, is given by

$$\langle q'' t'' | q' t' \rangle = N \int e^{\frac{i}{\hbar} S} \mathcal{D}q$$

where N is a normalization constant. Remember that to obtain this formula we need to assume $H = \frac{p^2}{2m} + V(q)$ (Ry 174) and the integral is performed over all paths such that $q(t') = q'$, $q(t'') = q''$.

More precisely, writing $|q' t'\rangle$ we are using the Heisenberg picture, but the transition amplitude of course does not depend on which picture we use, and indeed:

$$\langle q'' t'' | q' t' \rangle = \left({}_S\langle q''(t'') | \, e^{\frac{-i t'' H}{\hbar}} \right) \left(e^{\frac{i t' H}{\hbar}} |q'(t')\rangle_S \right)$$

$$= {}_S\langle q''(t'')| e^{-i\frac{(t''-t')H}{\hbar}} |q'(t')\rangle_S$$

which is a well-known formula for the transition amplitude (Sre, 57). The Heisenberg picture is more convenient for our purposes, so we won't write the subscript $|\rangle_H$ and all quantities will be in Heisenberg picture unless specified differently.

In general we want to compute expectation values of operators. Let us start with

$\langle q'' t'' | Q(t_1) | q' t'\rangle$ Inserting an identity:

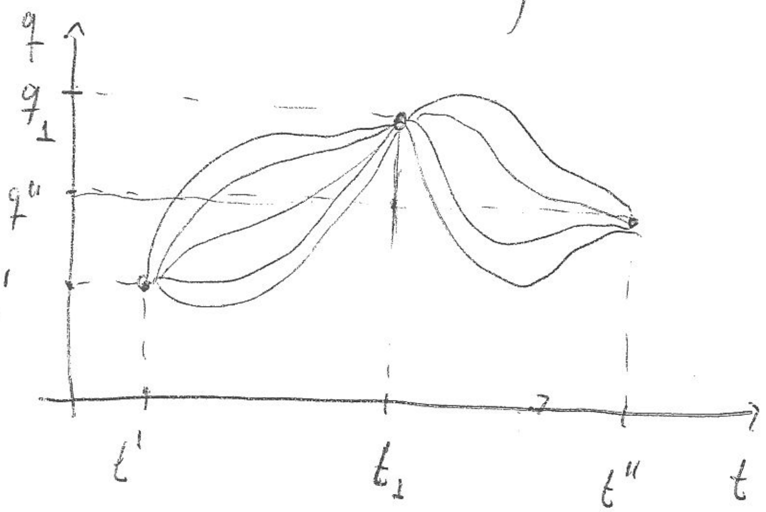
$$\langle q'' t'' | Q(t_1) \left(\int dq\, |q t_1\rangle\langle q t_1| \right) |q' t'\rangle =$$

$$\int dq\, \langle q'' t'' | \underbrace{Q(t_1)|q t_1\rangle}_{= q(t_1)|q t_1\rangle} \langle q t_1 | q' t'\rangle$$

$$= \int dq\, q(t_1) \langle q'' t'' | q t_1\rangle \langle q t_1 | q' t'\rangle$$

where now $q(t_1)$ is a scalar (not an operator)

Now $\langle q'' t'' | q t_1 \rangle$ is the amplitude for going to q'' at time t'' from q at time t_1; analogously $\langle q t_1 | q' t' \rangle$ is the amplitude for going from q' at time t' to q at time t_1. Hence $\langle q'' t'' | q t_1 \rangle \langle q t_1 | q' t' \rangle$ is interpreted as the amplitude to go from q' to q'' passing ~~through~~ through q at time t_1 (of course, we assume $t'' > t_1 > t'$):



But when we integrate over q we are allowing any path from q' to q'', hence $\int dq \, \langle q'' t'' | q t_1 \rangle \langle q t_1 | q' t' \rangle$ is the amplitude to go from q' at time t' to q'' at time t''.

Of course, in our case this amplitude is weighted by $q(t_1)$, hence

$$\langle q'' t'' | Q(t_1) | q' t' \rangle = N \int q(t_1)\, e^{\frac{i}{\hbar} S} \mathcal{D}q$$

(this can be computed explicitly, i.e, from $\int dq\, q(t_1) \langle \cdots \rangle$ discretize the time and apply all the steps leading to the path integral; the result is the above equality, Ry 177, St 66).

Now let us investigate expectation value of two operators. Our guess ~~and~~ would be

$$\langle q'' t'' | Q(t_1) Q(t_2) | q' t' \rangle = N \int q(t_1)\, q(t_2)\, e^{\frac{iS}{\hbar}} \mathcal{D}q$$

but we quickly see that this can not be the case: the operators $Q(t_1)$ and $Q(t_2)$ do not commute:

$$Q(t_1) Q(t_2) = e^{\frac{i t_1 H}{\hbar}} Q(0)\, e^{-\frac{i t_1 H}{\hbar}}\, e^{\frac{i t_2 H}{\hbar}} Q(0)\, e^{-\frac{i t_2 H}{\hbar}},$$

to move $Q(t_1)$ to the right we need to commute it with

$e^{\frac{it_2 H}{\hbar}}$, but $Q(0)$ and H don't commute since H involves the momentum operator.

On the other hand $q(t_1)$ and $q(t_2)$ are not operators, so they commute. The correct formula is

$$\langle q'' t'' | T(Q(t_1) Q(t_2)) | q' t' \rangle = N \int q(t_1) q(t_2) \, e^{\frac{iS}{\hbar}} \mathcal{D}q$$

where T is the time-ordering operator:

$$T(Q(t_1)Q(t_2)) = \begin{cases} Q(t_1)Q(t_2) & \text{if } t_1 > t_2 \\ Q(t_2)Q(t_1) & \text{if } t_1 < t_2 \end{cases}$$

Since by definition $T(Q(t_1)Q(t_2)) = T(Q(t_2)Q(t_1))$, we no longer have ordering issues and there is no contradiction with the fact that $q(t_1)$ and $q(t_2)$ are classical quantities.

To see that in fact we do get time ordering, consider again the idea of inserting identities:

$$\langle q'' t'' | Q(t_1)\, Q(t_2) | q' t' \rangle =$$

$$\langle q'' t'' | Q(t_1) \left(\int dq\, |q t_1\rangle \langle q t_1| \right) Q(t_2) \left(\int d\tilde{q}\, |\tilde{q} t_2\rangle \langle \tilde{q} t_2| \right) |q' t'\rangle =$$

$$\int\int dq\, d\tilde{q}\; q(t_1)\, \tilde{q}(t_2)\, \langle q'' t'' | q t_1\rangle \langle q t_1 | \tilde{q} t_2\rangle \langle \tilde{q} t_2 | q' t'\rangle$$

and as before we would like to interpret $\langle q'' t'' | q t_1\rangle \langle q t_1 | \tilde{q} t_2\rangle \langle \tilde{q} t_2 | q' t'\rangle$ as the amplitude to go from q' at time t' to q'' at time t'' passing ~~through~~ first through $\tilde{q}$ at time t_2 and then through q at time t_1 (since $\langle q t_1 | \tilde{q} t_2\rangle \langle \tilde{q} t_2 | q' t'\rangle$ means from q' to $\tilde{q}$ and then from $\tilde{q}$ to q).

But this obviously requires $t_2 < t_1$, so the path integral weighted by $q(t_1)$ and $q(t_2)$ corresponds to an amplitude with operators at earlier times appearing on the right — and this is exactly time ordering.

Of course, this ~~idea~~ idea generalizes to an arbitrary number of operators:

$$\langle q'' t'' | T(Q(t_1) \cdots Q(t_n)) | q' t' \rangle = N \int q(t_1) \cdots q(t_n) \, e^{\frac{iS}{\hbar}} \, \mathcal{D}q$$

<u>Caution</u> is common ~~to~~ physicists to write the above formula without the time-ordering T; it is assumed that it's "obvious" that the product should be time ordered.

Again, this could be seen directly, applying the steps leading to the path-integral (St 66)

We will see that this formula, generalized to the QFT setting, is a key part to compute scattering amplitudes.