4. Computing correlation functions

Given the correlation functions between vacuum states, the reduction formula tells us how to compute the amplitude probability. Recall that in the QM setting we showed that

$$\langle q'' t'' | q' t' \rangle = N \int_{\left\{ \begin{smallmatrix} q(t') = q' \\ q(t'') = q'' \end{smallmatrix} \right\}} e^{iS} \, \mathcal{D}q \qquad \text{and}$$

$$\langle q'' t'' | T(Q(t_1) \cdots Q(t_n)) | q' t' \rangle = N \int_{\left\{ \begin{smallmatrix} q(t') = q' \\ q(t'') = q'' \end{smallmatrix} \right\}} q(t_1) \cdots q(t_n) \, e^{iS} \, \mathcal{D}q .$$

The natural generalization to the QFT setting would be to replace the integral over paths by an integral over fields: $\mathcal{D}q \mapsto \mathcal{D}\varphi$, the product $q(t_1) \cdots q(t_n)$ by $\varphi(x_1) \cdots \varphi(x_n)$ ~~and the boundary conditions $q(t') = q'$, $q(t'') = q''$~~ (from a mathematical perspective, $\varphi(x_1)\ldots\varphi(x_n)$ is badly defined since fields are distributions). And since we want to use the reduction formula, for a matter of consistency we need to replace the boundary conditions $q(t')=q'$, $q(t'')=q''$ by boundary conditions which assure that we have the vacuum at $\pm\infty$ (since we assume this for deriving the reduction formula). It turns out that, differently from the QM case, it is too cumbersome to "pick" the fields satisfying the correct boundary conditions. Instead, it is easier to integrate over all fields and modify the Lagrangian in such a way that the corresponding state at $\pm\infty$. is the vacuum.

The trick consists of adding $\frac{1}{2}i\varepsilon\varphi^2$ to the Lagrangian, with $\varepsilon>0$ (eventually we'll take $\varepsilon\to 0$)

This produces an integrand of the form:
(using $\int L\,dt = \int\int \mathcal{L}\,d\vec{x}\,dt = \int \mathcal{L}\,dx$):

$$e^{i\int \mathcal{L}\,dx + i\int \frac{1}{2} i \frac{\varepsilon}{2} \varphi^2\,dx} = e^{iS}\, e^{-\frac{1}{2}\varepsilon\int \varphi^2\,dx}$$

It follows that fields which don't die off at infinity are damped by the exponential factor. So, in essence, the integrand behaves as if all fields vanish at infinity, i.e., instead of integrating only over the fields with the ~~approp~~ desired boundary conditions we integrate over all of them ~~and~~ introduce a factor which makes the integrand behave as if the field obeied the boundary conditions.

Of courses this is not totally convincing since the vacuum is defined by zero energy and not zero fields. We are not going to pursue this point here, but the argument can indeed be better elaborated (see Ry 175ff, St 69ff, Ze 22ff).

Anyway, it turns out that the ε term is necessary to make sense of some integrals which are going to arise. So if you prefer you may take the ε-prescription as a pure technical point.

So we obtain that in QFT the correlation functions are given as the path integral

$$\langle 0|T(\varphi(x_1)\cdots\varphi(x_n))|0\rangle = N\int \varphi(x_1)\cdots\varphi(x_n)\, e^{i\int(\mathcal{L}+\frac{i}{2}\varepsilon\varphi^2)dx}\,\mathcal{D}\varphi$$

Now we want to develop tools to compute the correlation functions. This is done by introducing a generating functional. Define:

$$Z[J] \equiv \int e^{i\int(\mathcal{L}+\varphi J+\frac{i}{2}\varepsilon\varphi^2)\,dx}\,\mathcal{D}\varphi$$

$J: \mathbb{R}^4 \to \mathbb{R}$ is a (compactly supported) function sometimes referred as source. Of course, when $J=0$

we simply recover the vacuum-to-vacuum amplitude:

$$\langle 0 | 0 \rangle = \int e^{i\int \mathcal{L} + \frac{i}{2}\epsilon\varphi^2} \mathcal{D}\varphi$$

Although J can be seen ~~as a~~ merely as a device to compute correlation functions (as we show below), we can give it a physical interpretation, as follows. The process we are interested in involves the creation of a particle somewhere in spacetime, its propagation and interaction and further annihilation. The act of creation may be represented by a source and the annihilation by a sink, which is in a manner of speaking, a source. (Ry 174, Ze 20). Introducing a source is something we know how to do from classical field theory: we add a term proportional to $J(x)\varphi(x)$ to the Lagrangian: for example, ~~in~~ in free case $\mathcal{L} = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 + J\varphi$, giving a dynamical equation: $(\Box + m^2)\varphi(x) = J(x)\varphi(x)$. The source J plays a role analogous to that of the electromagnetic current, which acts as a "source" for the electromagnetic field — the generalization of this idea to the QFT setting is due to Schwinger (Ry 173). In the end, after doing the calculations, we will set the source to zero, what may be interpreted as the fact that in $\pm\infty$ the source is zero (that's why we require compact support).

Let us start with the Lagrangian of the free field theory. We denote the corresponding generating functional $Z_0[J]$:

$$Z_0[J] = \int e^{i\int \frac{1}{2}\left(\partial_\mu\varphi\partial^\mu\varphi - (m^2 - i\varepsilon)\varphi^2 + J\varphi\right)} \mathcal{D}\varphi$$

We can integrate by parts on the exponential to get (ignoring boundary terms)

$$\int \partial_\mu\varphi\partial^\mu\varphi \, dx = -\int \varphi\Box\varphi \, dx \quad \text{so that}$$

$$Z_0[J] = \int e^{-i\int \frac{1}{2}(\varphi(\Box + m^2 - i\varepsilon)\varphi - \varphi J)\,dx}\,\mathcal{D}\varphi$$

remark: we do not have $(\Box + m^2)\varphi = 0$ since φ does not obey Klein-Gordon (we integrate over all the fields).

~~the~~ To evaluate the above expression, consider a solution $(\Box + m^2 - i\varepsilon)\varphi_0 = J$, and then ~~perform~~ perform a translation $\varphi \mapsto \varphi + \varphi_0$ (the measure $\mathcal{D}\varphi$ is translation invariant) and use integration by parts again: $\int \varphi_0(\Box + m^2 - i\varepsilon)\varphi = \int \varphi(\Box + m^2 - i\varepsilon)\varphi_0$ so that:

$$\int \left(\frac{1}{2}\varphi(\Box + m^2 - i\varepsilon)\varphi - \varphi J\right)dx \mapsto \int \Big(\frac{1}{2}\varphi(\Box + m^2 - i\varepsilon)\varphi + 2\cdot\frac{1}{2}\varphi\underbrace{(\Box + m^2 - i\varepsilon)\varphi_0}_{=J}$$

$$+ \frac{1}{2}\varphi_0\underbrace{(\Box + m^2 - i\varepsilon)\varphi_0}_{=J} - \varphi J - \varphi_0 J\Big)\,dx$$

$$= \int \frac{1}{2}\varphi(\Box + m^2 - i\varepsilon)\varphi - \frac{1}{2}\varphi_0 J$$

Now let us find an expression for φ_0.

It is given by

$$\varphi_0(x) = -\int \Delta(x-y) J(y)\, dy$$

where $\Delta(x)$ is the fundamental solution

$$(\Box + m^2 - i\varepsilon)\Delta(x) = -\delta(x)$$

Δ is also called propagator or Feynmann propagator (we'll discuss its physical interpretation later)

So we can write:

$$\int \left(\frac{1}{2}\varphi(\Box + m^2 - i\varepsilon)\varphi - \frac{1}{2}\varphi_0 J\right) dx = \frac{1}{2}\int \varphi(\Box + m^2 - i\varepsilon)\varphi\, dx +$$

$$+ \frac{1}{2}\int J(x)\,\Delta(x-y)\,J(y)\, dx\, dy$$

Putting this into the expression for $Z_0[J]$:

$$Z_0[J] = \int e^{-i\int \frac{1}{2}\varphi(\Box + m^2 - i\varepsilon)\varphi - \varphi J\, dx} \mathcal{D}\varphi =$$

$$\int e^{-\frac{i}{2}\int \varphi(\Box + m^2 - i\varepsilon)\varphi\, dx} \cdot e^{-\frac{i}{2}\int J(x)\,\Delta(x-y)\,J(y)\, dx\, dy} \mathcal{D}\varphi$$

The second exponential does not involve φ and so it can be pulled out of the integration over φ:

$$Z_0[J] = \left(\int e^{-\frac{i}{2}\int \varphi(\Box + m^2 - i\varepsilon)\varphi\, dx} \mathcal{D}\varphi \right) e^{-\frac{i}{2}\int J(x)\Delta(x-y)J(y)\,dx\,dy}$$

$$Z_0[J] = N\, e^{-\frac{i}{2}\int J(x)\Delta(x-y)J(y)\,dx\,dy} \qquad (*)$$

The integral over φ is a number which we called N. Since we will be interested in normalized transition amplitudes the value of N is irrelevant. In particular, notice that we no longer have to compute a path-integral.

One important remark: Δ has a Fourier representation as

$$\Delta(x) = \frac{1}{(2\pi)^4}\int \frac{e^{-ikx}}{k^2 - m^2 + i\varepsilon}\, dk$$

Here the necessity of the ε term becomes clear. So, as we said, the ε prescription can be considered as a technical point used to make sense of some integrals.

Finally we set the normalization by $N=1$. This is equivalent to redefine Z_0 as $Z_0[J]/Z_0[0]$ (Ry 191)

More precisely:

$$Z_0[J] = \frac{\int e^{-i\int\left(\frac{1}{2}\varphi(\Box+m^2-i\epsilon)\varphi \;-\; J\varphi\right)dx}\,\mathcal{D}\varphi}{\int e^{-i\int \frac{1}{2}\varphi(\Box+m^2-i\epsilon)\varphi}\,\mathcal{D}\varphi} \qquad \circledast$$

So, after doing the steps leading to ($\star$) on p. 44 we find exactly:

$$Z_0[J] = e^{-\frac{i}{2}\int J(x)\,\Delta(x-y)\,J(y)\,dx\,dy} \qquad \circledast\circledast$$

since the denominator in our new definition is precisely N.

Notice that $Z_0[0] = 1$, ~~which physically means the normalization~~ ~~$=1$~~.

Let us compute a correlation function. Let us do $\langle 0|\varphi(y)\,\varphi(x)|0\rangle$, also known as 2-point function. (In general $\langle 0|T(\varphi(x_1)\dots\varphi(x_n))|0\rangle$ is the <u>n-point function or Green function</u>). Notice that we use physicists abuse of notation and supress $T(\;\;)$.

Recall the functional derivative:

$$\frac{\delta}{\delta J(z)} F[J] = \lim_{\varepsilon \to 0} \frac{F[J(x) + \varepsilon\, \delta(x-z)] - F[J(x)]}{\varepsilon}$$

(see the appendix for the mathematical content of this expression).

Now notice that each time we take a functional derivative of we have a φ coming down:

$$\frac{\delta}{\delta J(x_1)} Z_0[J] = \frac{\int i\varphi(x_1)\, e^{-i\int(\frac{1}{2}\varphi(\Box + m^2 - i\varepsilon)\varphi - J\varphi)\,dx}\, \mathcal{D}\varphi}{\int e^{-i\int \frac{1}{2}\varphi(\Box + m^2 - i\varepsilon)\varphi - \cancel{J\varphi}\,dx}\, \mathcal{D}\varphi}$$

$$\frac{\delta}{\delta J(x_2)} \frac{\delta}{\delta J(x_1)} Z_0[J] = \left(\int i^2 \varphi(x_1)\varphi(x_2)\, e^{-\int(\ )}\, \mathcal{D}\varphi\right) \Big/ \int e^{-i\int(\ )}\, \mathcal{D}\varphi$$

If we proceed in this way and set $J=0$:

$$\frac{1}{i^n} \left. \frac{\delta^n Z_0[J]}{\delta J(x_1) \ldots \delta J(x_n)} \right|_{J=0} = \frac{\int \varphi(x_1) \cdots \varphi(x_n)\, e^{-i\int \frac{1}{2}\varphi(\Box + m^2 - i\varepsilon)\varphi\, dx}\, \mathcal{D}\varphi}{\int e^{-i\int \frac{1}{2}\varphi(\Box + m^2 - i\varepsilon)\varphi\, dx}\, \mathcal{D}\varphi}$$

Now we claim that the above expression is exactly the correlation function we want to compute. At first glance it appears that it is the correlation divided by the vacuum-to-vacuum amplitude $\langle 0|0\rangle = \int e^{-iS(\,)} \mathcal{D}\varphi$. But recall that ~~the~~ (i) when introducing the correlation functions we always had a normalization constant that we haven't taken care of; (see p. 37)
(ii) in most computations (including reduction formula) we are assuming a normalization $\langle 0|0\rangle$, which is not the case here. So, returning to p. 37 we should choose N such that

$$\langle 0|T(\varphi(x_1)\cdots\varphi(x_n))|0\rangle = \frac{\int \varphi(x_1)\cdots\varphi(x_n)\, e^{iS(\,)}\,\mathcal{D}\varphi}{\int e^{i(\,)}\,\mathcal{D}\varphi}$$

So we do have:

$$\langle 0|T(\varphi(x_1)\cdots\varphi(x_n))|0\rangle = \frac{1}{i^n}\frac{\delta^n Z_0[J]}{\delta J(x_1)\cdots\delta J(x_n)}$$

Now, the importan ~~think~~ thing is that, on the other hand, $Z_0[J]$ is given by ●●, which can be used to compute explicitely the correlation functions:

$$\frac{\delta Z_0[J]}{\delta J(x_1)} = \frac{\delta}{\delta J(x_1)} e^{-\frac{i}{2}\int J(x)\,\Delta(x-y)\,J(y)\,dx\,dy} =$$

$$= e^{-\frac{i}{2}\int J(x)\,\Delta(x-y)\,J(y)\,dx\,dy} \frac{\delta}{\delta J(x_1)}\left(-\frac{i}{2}\int J(x)\,\Delta(x-y)\,J(y)\,dy\,dx\right)$$

$$= e^{(\ )}\left(-\frac{i}{2}\int\left(\frac{\delta J(x)}{\delta J(x_1)}\,\Delta(x-y)\,J(y)\,dx\,dy + J(x)\,\Delta(x-y)\,\frac{\delta J(y)}{\delta J(x_1)}\right)dx\,dy\right)$$

Using $\frac{\delta J(x)}{\delta J(y)} = \delta(x-y)$:

$$= -\frac{i}{2}\,e^{(\ )}\left[\left(\int \Delta(x_1-y)\,J(y)\,dy\right) + \left(\int J(x)\,\Delta(x-x_1)\,dx\right)\right]$$

Using $\Delta(x) = \frac{1}{(2\pi)^4}\int \frac{e^{-ikx}}{k^2-m^2+i\varepsilon}\,dk$, we see, changing $k \mapsto -k$, that

$$\Delta(x_1-y) = \Delta(y-x_1)$$

So

$$\frac{\delta Z_0[J]}{\delta J(x_1)} = -i\, e^{-\frac{i}{2}\int J(x)\Delta(x-y)J(y)\,dx\,dy} \int \Delta(x-x_1)\, J(x)\, dx$$

Computing one more derivative (Ry (92)):

$$\frac{\delta}{\delta J(x_2)} \frac{\delta}{\delta J(x_1)} Z_0[J] = i^3 \Delta(x_1 - x_2)\, e^{-\frac{i}{2}\int J(x)\Delta(x-y)J(y)\,dx\,dy}$$

$$+ \int \Delta(x-x_1)\, J(x_1)\, dx_1 \int \Delta(x_2 - x_1)\, J(x_1)\, dx_1\, e^{-\frac{i}{2}\int (\qquad)\,dx\,dy}$$

plugging $J=0$ we obtain:

$$\langle 0 | T(\varphi(x_1)\varphi(x_2)) | 0 \rangle = \frac{1}{i^2} \frac{\delta^2 Z_0[J]}{\delta J(x_1)\delta J(x_2)} = i\Delta(x_1 - x_2)$$

So we can go back to the reduction formula and compute the amplitude; instead of x_1 and x_2 we write x and y and instead of $k, \tilde{k}$ we write k and q.

Recall the reduction formula from p. 34:

$$\langle f | i \rangle = i^2 \int dx\,dy\, e^{iqy} (\Box_y + m^2)\, e^{-ikx} (\Box_x + m^2) \overbrace{\langle 0 | T(\varphi(x)\varphi(y)) | 0 \rangle}^{= i\Delta(x-y)}$$

$$= i^3 \int dx\,dy\, e^{iqy} e^{-ikx} (\Box_y + m^2)(\Box_x + m^2)\, \Delta(x-y)$$

Recall that the propagator is the fundamental solution $(\Box + m^2)\Delta(x) = \delta(x)$

so $(\Box_x + m^2)\Delta(x-y) = \delta(x-y)$ and hence

$$\langle f | i \rangle = i^3 \int dx\, dy\; e^{iqy} e^{-ikx} (\Box_y + m^2)\,\delta(x-y)$$

We can integrate by parts the term $\Box_y \delta(x-y)$

to get

$$\langle f | i \rangle = i^3 \int dx\, dy \Big[\underbrace{\left(\Box_y e^{iqy}\right)}_{= -q^2 e^{iqy}} e^{-ikx} \delta(x-y) + e^{iqy} e^{-ikx} m^2 \delta(x-y) \Big]$$

$$= i^3 \int dx\, dy\, (-q^2 + m^2)\, e^{iqy} e^{-ikx} \delta(x-y) = 0$$

since $q^2 = m^2$. Of course, this was expected since the probability of starting with a particle of momentum k and end up with momentum q is zero because there is no interaction (recall that in deducing the reduction formula we assume $q \neq k$).

We can compute more n-point functions.

If n is odd, then we'll take and odd number of functional derivatives of $e^{-\frac{i}{2}\int J(x)\,\Delta(x-y)\,J(y)\,dx\,dy}$ and there will be always a remaining J which, when set to zero, kills the whole expression. So

$\langle 0|\, T(\varphi(x_1)\cdots\varphi(x_{2n+1}))\,|0\rangle = 0$, what is expected, since the number of particles remains constant in a free theory.

For n even, an easy but lengthly calculation shows that

$$\langle 0|T(\varphi(x_1)\cdots\varphi(x_{2n}))|0\rangle = \sum_{\text{permutations}} \langle 0|T(\varphi(x_{p_1})\varphi(x_{p_2}))|0\rangle \cdots \langle 0|T(\varphi(x_{p_{2n-1}})\varphi(x_{p_{2n}}))|0\rangle$$

where, as computed before, $\langle 0|T(\varphi(x)\varphi(y))|0\rangle = i\,\Delta(x-y)$. The calculation can be found in R, p. 195.