5. Correlation functions for the interacting theory and the perturbative expansion

Consider now an interacting theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4 = \mathcal{L}_0 + \mathcal{L}_I$$

Recall from p. 37 that we want to compute (with our new normalization)

$$\langle 0|T(\varphi(x_1)\cdots\varphi(x_n))|0\rangle = \frac{\int \varphi(x_1)\,\varphi(x_2)\cdots\varphi(x_n)\, e^{i\int(\mathcal{L}+\frac{i}{2}\varepsilon\varphi^2)\,dx}\,\mathcal{D}\varphi}{\int e^{i\int(\mathcal{L}+\frac{i}{2}\varepsilon\varphi^2)\,dx}\,\mathcal{D}\varphi}$$

As before, we introduce a generating functional

$$Z[J] = \left(\int e^{i\int(\mathcal{L}+\frac{i}{2}\varepsilon\varphi^2+J\varphi)\,dx}\,\mathcal{D}\varphi\right)\Big/\left(\int e^{i\int(\mathcal{L}+\frac{i}{2}\varepsilon\varphi^2)\,dx}\right)$$

And we can obtain the correlation functions by differentiating $Z[J]$ and setting $J=0$. As we did in the free case, we are going to derive a useful formula for doing this computation.

Let us first do this in a baby model (here I follow Ze p. 41 ff). Consider the one dimensional integral:

$$Z(J) = \int_{-\infty}^{+\infty} e^{-\frac{1}{2}m^2 q^2 - \frac{\lambda}{4!} q^4 + Jq} \, dq$$

$$= \int_{-\infty}^{+\infty} e^{-\frac{1}{2}m^2 q^2 + Jq} \left[1 - \frac{\lambda}{4!} q^4 + \frac{1}{2} \left(\frac{\lambda}{4!}\right)^2 q^8 + \dots \right]$$

Now we want to integrate term by term. The trick is to write:

$$\int_{-\infty}^{+\infty} q^{4n} e^{-\frac{1}{2}m^2 q^2 + Jq} \, dq = \frac{d^{4n}}{dJ^{4n}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}m^2 q^2 + Jq} \, dq$$

The Gaussian integral can be performed and

$$= \frac{d^{4n}}{dJ^{4n}} \sqrt{\frac{2\pi}{m^2}} \, e^{\frac{J^2}{2m^2}} .$$

Hence

$$Z(J) = \sqrt{\frac{2\pi}{m^2}} \, e^{-\frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4} \, e^{\frac{1}{2m^2} J^2}$$

Notice that $\sqrt{\frac{2\pi}{m^2}} = Z(0)$, ~~if we~~ redefine

~~$Z(J)$ as $\frac{Z(J)}{Z(0)}$ as we did before we can~~

~~drop the factor $\sqrt{2\pi/m^2}$~~

Notice that now we can compute any term in the expansion of $Z(J)$. For example, suppose we want the term of order λJ^4. We do this by expanding the two exponentials $e^{-\frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4}$ and $e^{\frac{1}{2m^2}J^2}$ and matching the correct powers. This is easy: first, λ appears only on the first exponential, so the (only) term linear in λ is $-\frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4$. This term acts on $e^{\frac{1}{2m^2}J^2} = 1 + \frac{1}{2m^2}J^2 + \frac{1}{2!}\left(\frac{1}{2m^2}\right)^2 J^4 + \dots$

Since we act with four derivatives, $\frac{d^4}{dJ^4}$, and we want λJ^4 we need to keep the term in J^8, i.e.,

$\cancel{\frac{1}{4!}\left(\frac{1}{2m^2}\right)}$ $\frac{1}{4!}\left(\frac{1}{2m^2}\right)^4 (J^2)^4$ so the term in λJ^4 is

$$-\frac{\lambda}{4!}\,\frac{1}{4!}\left(\frac{1}{2m^2}\right)^4 8\cdot 7\cdot 6\cdot 5\,\lambda J^4 = -\lambda\,\frac{1}{(4!)^3}\,8!\left(\frac{1}{2m^2}\right)^4 J^4$$

As another example we see that the term in $\lambda^2 J^6$ is

$$\frac{1}{2}\left(-\frac{\lambda}{4!}\right)^2 \frac{d^8}{dJ^8}\left(\frac{1}{7!}\left(\frac{1}{2m^2}\right)^7 (J^2)^7\right) =$$

$$= \frac{1}{2}\lambda^2 \frac{1}{4!}\frac{1}{7!}\frac{1}{6!}\, 14! \left(\frac{1}{2m^2}\right)^7 J^6$$

(all this up to the term $\sqrt{\frac{2\pi}{m^2}}$)

We will have more to say about this way of collecting the terms when we explore the Feynmann diagrams.

As a preparation to the QFT setting, let us do the same with a d-dimensional integral:

$$Z(J) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{+\infty} dq_1 \cdots dq_d \; e^{-\frac{1}{2}qAq - \frac{\lambda}{4!}q^4 + J\cdot q}$$

$J\cdot q$ = dot product

($q^4 = \sum_i q_i^4$). A is the matrix of a (non-degenerate) quadratic form (the natural way of generalizing ~~the~~ m^2q^2). Repeating the argument we obtain:

$$Z(J) = \sqrt{\frac{(2\pi)^d}{\det A}}\; e^{-\frac{\lambda}{4!}\sum_i \frac{\partial}{\partial J_i}}\; e^{\frac{1}{2}J\cdot A^{-1}J}$$

How do we generalize this formula to the QFT setting?

The partial ~~dx~~ derivatives become functional derivatives $\frac{\delta}{\delta J(z)}$. What is the quadratic formula A? The quadratic form corresponds to the free part in the finite dimensional integral. In the QFT case, the free part is

$$\frac{1}{2}\, \partial^\mu \varphi\, \partial_\mu \varphi - (m^2 - i\varepsilon)\varphi^2$$

which we showed (p. 39, 40) that can be integrated by parts to give

$$\frac{1}{2}\, \varphi\,(\Box + m^2 - i\varepsilon)\,\varphi$$ so we have the correspondence

$\frac{1}{2}A \longmapsto \frac{1}{2}(\Box + m^2 - i\varepsilon)$ (the $\int$ in $\frac{1}{2}\int \varphi(\ )\varphi$ is the generalization of the sum in the finite dimensional case: $qAq = A_{ij}q^i q^j$). But we need the inverse of the quadratic form. In the continuum case the role of the inverse is played by the fundamental solution, i.e., by the propagator

$$(\square + m^2 - i\varepsilon)\,\Delta(x) = \delta(x)$$

Since this is the continuous analogous of $A_{ij}A^{jk} = \delta_i^k$

So we may write $e^{-\frac{i}{2}\int J(x)\Delta(x-y)J(y)\,dx\,dy}$ in place of $e^{\frac{1}{2}JA^{-1}J}$. Putting everything together

$$\int e^{i\int \mathcal{L} + \frac{i}{2}\varepsilon\varphi^2 + J\varphi}\,\mathcal{D}\varphi = \mathcal{C}\, e^{-\frac{i\lambda}{4!}\int dz\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4} e^{-\frac{i}{2}\int J(x)\Delta(x-y)J(y)\,dx\,dy}$$

where $\mathcal{C}$ is the analogous of $\sqrt{\frac{(2\pi)^d}{\det A}}$, so it corresponds to the integral of the free theory with no J: $\mathcal{C} = \int e^{i\int \mathcal{L}_0 + \frac{i}{2}\varepsilon\varphi^2}\mathcal{D}\varphi$. Therefore (from p. 50)

$$Z[J] = \left(\int e^{i\int \mathcal{L} + \frac{i}{2}\varepsilon\varphi^2 + J}\,\mathcal{D}\varphi\right) \Big/ \left(\int e^{i\int\mathcal{L} + \frac{i}{2}\varepsilon\varphi^2}\,\mathcal{D}\varphi\right)$$

$$= \left(\int \qquad\right) \Big/ \left(\left(\int e^{i\int\mathcal{L} + \frac{i}{2}\varepsilon\varphi^2 + J}\,\mathcal{D}\varphi\right)\right)\Bigg|_{J=0}$$

$$= \frac{\cancel{\mathcal{G}}\, e^{-\frac{i\lambda}{4!}\int dz \left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4} e^{-\frac{i}{2}\int J(x)\Delta(x-y)J(y)\,dx\,dy}}{\cancel{\mathcal{G}}\, e^{-\frac{i\lambda}{4!}\int dz \left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4} e^{-\frac{i}{2}\int J(x)\Delta(x-y)J(y)\,dx\,dy}\Big|_{J=0}}$$

So

$$Z[J] = \frac{e^{-\frac{i\lambda}{4!}\int dz \left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4} e^{-\frac{i}{2}\int J(x)\Delta(x-y)J(y)\,dx\,dy}}{\left[e^{-\frac{i\lambda}{4!}\int dz \left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4} e^{-\frac{i}{2}\int J(x)\Delta(x-y)J(y)\,dx\,dy}\right]_{J=0}}$$

Some remarks: First, there was a difference in the form of the integral we wanted to compute from ~~&~~ its finite dimensional analogous, which is the presence of i. So we supposed that the integrals were Wick-rotated.

The factor $\frac{1}{i}$ is to compensate for i in the $e^{-\frac{i}{2}}$ expression.

Second, all this construction by analogy with the finite dimensional integral may sound sloppy, but if you redo the steps of the finite dimensional case in the infinite dimensional one you'll see that all we are ~~the~~ using are properties of integrals and functional derivatives which, at least at the formal level, are the same of finite sums and partial derivatives. Anyway, a less "tricky" derivation of the formula for $Z[J]$ may be found on Ry p. 200 ff.

Finally, it is possible to show that an analogous formula holds in general, i.e, with $e^{i\int \mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)dz}$ in place of $e^{\frac{-i\lambda}{4!}\int\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4 dz}$, see Ry 200 ff.

~~Let us review what we have accomplished. We want to compute the correlation functions:~~

Let us compute to first order the two point correlation function, so we need to compute $\frac{1}{i^2}\frac{\delta^2}{\delta J(x_a)\,\delta J(x_b)} Z[J]\Big|_{J=0}$

where $Z[J] = \dfrac{e^{-\frac{i\lambda}{4!}\int dz\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4} e^{-\frac{i}{2}\int J(x)\Delta(x-y)J(y)}}{\left[e^{-\frac{i\lambda}{4!}\int dz\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4} e^{-\frac{i}{2}\int J(x)\Delta(x-y)J(y)}\right]_{J=0}}$

Let us start with the denominator. Since ~~we are~~ on it we set $J=0$, we keep only terms with no J. So, expanding both exp:

$$\left(1 - \frac{i\lambda}{4!}\int dz \frac{1}{i^4}\frac{\delta^4}{\delta J(z)^4} + \dots\right)\left(1 + \dots + \frac{1}{2!}\left(\frac{-i}{2}\right)^2\left(\int J(x)\Delta(x-y)J(y)\right)^2 + \dots\right]$$

We need the term of order 2 on the second ~~[illegible]~~ exponential because it contains 4 J's and we'll act with four derivatives. So, multiplying and keeping only terms of interest:

$$1 - \frac{i\lambda}{4!}\left(-\frac{1}{4}\right)\frac{1}{2!}\int dz\,\frac{1}{i^4}\frac{\delta^4}{\delta J(z)^4}\overbrace{\int\cdots\int}^{4\text{ integrals}} J(x_1)J(x_2)J(x_3)J(x_4)\,\Delta(x_1-x_2)\,\Delta(x_3-x_4)$$

The only derivatives which survive when $J=0$ are those with one derivative for each J. The first derivative can hit any of the J's, so we have

$\frac{\delta J(x_1)}{\delta J(z)} J(x_2) J(x_3) J(x_4)$, or $J(x_1) \frac{\delta J(x_2)}{\delta J(z)} J(x_3) J(x_4)$ etc.

After relabeling (x_i are dummy) we have

$4 \frac{\delta J(x_1)}{\delta J(z)} J(x_2) J(x_3) J(x_4)$. Now the second derivative will be acting on the three remaining J's and produce

$$4 \cdot 3 \cdot \frac{\delta J(x_1)}{\delta J(z)} \frac{\delta J(x_2)}{\delta J(z)} J(x_3) J(x_4).$$ In the end:

$$4! \frac{\delta J(x_1)}{\delta J(z)} \cdots \frac{\delta J(x_4)}{\delta J(z)} = 4! \, \delta(x_1 - z) \, \delta(x_2 - z) \, \delta(x_3 - z) \, \delta(x_4 - z)$$

So we obtain for the denominator:

$$1 - \frac{i\lambda}{4!} \left(-\frac{1}{4}\right) \frac{1}{2!} \int dz \; 4! \, \Delta(z-z) \, \Delta(z-z) + \text{terms in } J \Bigg]_{J=0}$$

$$= 1 - \frac{i\lambda}{4!} \int (-3) \Delta^2(0) \, dz.$$ Using the binomial:

$$\frac{1}{1 - \frac{i\lambda}{4!} \int (-3) \Delta^2(0) \, dz} = 1 + \frac{i\lambda}{4!} \int (-3) \Delta^2(0) \, dz + \mathcal{O}(\lambda^2)$$

For the numerator we need to keep terms with 2 J's (since in the end we take 2 derivatives of $Z[J]$ and set $J=0$). Up to first order in λ:

$$\frac{\left(1 - \frac{i\lambda}{4!}\int dz \left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4 + \cdots\right) e^{-\frac{i}{2}\int J \Delta J}}{1 - \frac{i\lambda}{4!}\int (-3)\Delta^2(0)\, dz} =$$

$$\left(1 - \frac{i\lambda}{4!}\int dz \left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4 + \cdots\right)\left(1 + \frac{i\lambda}{4!}\int (-3)\Delta^2(0)\, dz + \cdots\right) e^{-\frac{i}{2}\int J\Delta J}$$

So the term in λ is

$$\frac{i\lambda}{4!}\left[\int (-3)\Delta^2(0)\, dz - \int dz \left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4\right] e^{-\frac{i}{2}\int J(x)\Delta(x-y)J(y)}$$

Now we expand the other exp. and keep the terms with 2 J's. That means that $\int (-3)\Delta^2(0)\, dz$ goes with the term $-\frac{i}{2}\int J(x)\Delta(x-y)J(y)$ and $\int dz \left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4$ with a term containing 6 J's, i.e, with $\frac{1}{3!}\left(-\frac{i}{2}\right)^3\left[\int J(x)\Delta(x-y)J(y)\right]^3$. Let us treat this last term first:

$$\frac{1}{3!}\left(\frac{-i}{2}\right)^3 \frac{1}{i^4}\int dz \frac{\delta^4}{\delta J(z)^4} \underbrace{\int\cdots\int}_{6\ \text{integrals}} J(x_1)\cdots J(x_6)\,\Delta(x_1-x_2)\,\Delta(x_3-x_4)\,\Delta(x_5-x_6)$$

Again, the relevant terms (ie, resulting in 2 J's) are those with one derivative at each different J.

Basically there are two types of such terms: those whose terms with no derivatives are a pair whose coordinates appear in the same propagator and those which ~~ap~~ ~~appearance~~ such as $J(x_1)\,J(x_2)\,\frac{\delta J(x_3)}{\delta J(z)}\cdots\frac{\delta J(x_6)}{\delta J(z)}$

(so both variables in $\Delta(x_1-x_2)$ are not with derivative, in the corresponding J's); and those whose terms with no derivatives correspond to ~~the~~ different propagators, such as $J(x_1)\,\frac{\delta J(x_2)}{\delta J(z)}\,J(x_3)\,\frac{\delta J(x_4)}{\delta J(z)}\cdots\frac{\delta J(x_6)}{\delta J(z)}$

For the first type we ~~may~~ have:

$\underbrace{x_1\ x_2}_{\text{no } \frac{\delta}{\delta J}}$ $\underbrace{x_3\ x_4\ x_5\ x_6}_{\frac{\delta}{\delta J(z)} \text{ in all possible ways} = 4!}$, the same with $x_3\ x_4$ $\underbrace{\quad}$ and $x_5\ x_6$ $\underbrace{\quad}$

So we have (after relabeling dummy)

$$\Big(\quad\Big) 3\times 4! \int dz \int\!\cdots\!\int J(x_1)\, J(x_2)\, \delta(x_3-z)\,\delta(x_4-z)\,\delta(x_5-z)\,\delta(x_6-z)$$
$$\Delta(x_1-x_2)\,\Delta(x_3-x_4)\,\Delta(x_5-x_6)$$

$$= \frac{1}{3!}\left(\frac{-i}{2}\right)^3 3\times 4! \int J(x_1)\,\Delta(x_1-x_2)\,J(x_2)\,\Delta^2(0)$$

$$= \Delta^2(0)\,\frac{3i}{2}\int dz\, J(x_1)\,\Delta(x_1-x_2)\,J(x_2)$$

For the second type we have some similar combinatorics, but an extra factor of 2×2: above, fixing $J(x_1)$ as with no $\frac{\delta}{\delta J(z)}$ determines that the other with no $\frac{\delta}{\delta J(z)}$ is $J(x_2)$, but now it may be $J(x_3)$ or $J(x_5)$, since both produce ~~de~~ ~~from~~ the same term (after relabeling). Then repeat the idea fixing x_2 (which goes with x_3 or x_5 fixed) (*)

$$\Big(\quad\Big) 2\times 2\times 3\times 4! \int dz\; J(x_1)\,J(x_2)\,\Delta(x_1-z)\,\Delta(x_2-z)\,\Delta(0)$$

$$= \Delta(0)\; 6i \int dz\; J(x_1)\,J(x_2)\,\Delta(x_1-z)\,\Delta(x_2-z)$$

Putting all together, the term in order λ is

(*) before we didn't consider the case fixing x_1 and x_2 separately cause one determined the other.

$$\frac{i\lambda}{4!}\Bigg[\cancel{\int (-3)\,\Delta^2(0)\,dz \;\left(\frac{-i}{2}\right)\int J(x)\,\Delta(x-y)\,J(y)} \; -$$

$$- \cancel{\Delta^2(0)\,\frac{3i}{2}\int dz \; J(x_1)\,\Delta(x_1-x_2)\,J(x_2)}$$

$$- \Delta(0)\,6i \int dz \; J(x_1)\,J(x_2)\,\Delta(x_1-z)\,\Delta(x_2-z) \Bigg]$$

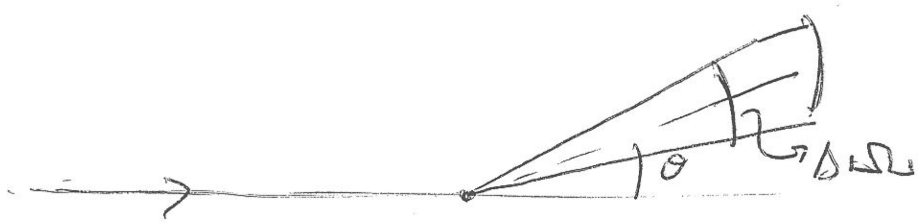
(the fact that the terms in $\Delta^2(0)$ cancel is not coincidence, it's a property of normalized J's).

$$= \frac{\lambda}{4}\int dz \; J(x_1)\,J(x_2)\,\Delta(x_1-z)\,\Delta(x_2-z)\,dx_1\,dx_2$$

Finally, taking $-\dfrac{\delta^2}{\delta J(x_a)\,\delta J(x_b)}$ (and consider the factor of two cause we may combine x_1, x_a or x_1, x_b):

$$= -\frac{\lambda}{2}\,\Delta(0)\int \Delta(z-x_a)\,\Delta(z-x_b)\,dz$$

Physical scattering: beam of ctc energy sent toward the target, beam of ctc wide spread and uniform density ρ of particles per unit of area of the plane $\perp$ to the beam. Detector sits at some scattering angle $\langle \theta, \varphi \rangle$ and counts all particles that leave the target within some angular region (solid angle) $\Delta\Omega$ about (θ,φ):



The measured quantity is

$$\frac{\#\ \text{particles hitting detector}}{(\Delta\Omega)}$$

Cross section : express the probability of interaction between particles.

Classically: a "beam" of (a large # of) ~~part~~ point particles is shot into a solid target. Interaction occurs with probability of 100% if the particle hits the solid and ~~not~~ doesn't occur (0%) if it doesn't hit then the interaction probability is ~~the~~ simply the ratio of the area of the section of the target to the total target area. This basic idea is then extended to the case where the target is not homogeneous / mediated by a non-homogeneous field and then we have probabilities between 0 and 100%.