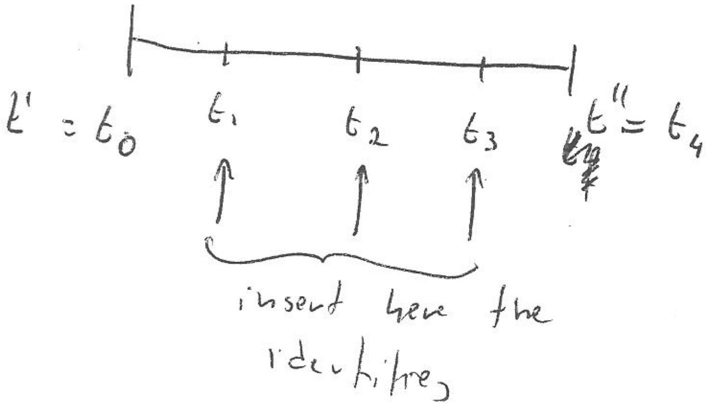
Appendix

1. Path integral

The derivation of the path integral is a standard one and can be found in any book, here we simply give the general ~~[illegible]~~ idea

We want to compute $\langle q''t''|q't'\rangle$. Divide the interval $t''-t'$ in $n+1$ pieces: $\tau = \frac{t''-t'}{n+1}$. Then at each t_i, $i=1,\dots,n$, insert an identity of the form $\int dq\,|q_it_i\rangle\langle q_it_i|$; notice that we have n such pts for $\tau = \frac{t''-t'}{n+1}$ (ie, for $n+1$ pieces), for example $n+1=4$



So

$$\langle q'' t'' | q' t' \rangle = \int \cdots \int dq_1\, dq_2 \cdots dq_n \, \langle q'' t'' | q_n t_n \rangle \langle q_n t_n | q_{n-1} t_{n-1} \rangle \cdots$$
$$\cdots \langle q_1 t_1 | q' t' \rangle .$$

But $\langle q_{j+1} t_{j+1} | q_j t_j \rangle = \langle q_{j+1} | e^{\frac{-i\tau H}{\hbar}} | q_j \rangle$

since $| q_{j+1} t_{j+1} \rangle = e^{\frac{i t_{j+1} H}{\hbar}} | q \rangle$, $t_{j+1} = t_j + \tau$ and $| q_j t_j \rangle = e^{\frac{i t_j H}{\hbar}} | q_j \rangle$. Since τ is small we can expand and keep only first order

$$= \langle q_{j+1} | 1 - \frac{i}{\hbar} \tau H | q_j \rangle = \underbrace{\delta(q_{j+1} - q_j)}_{= \frac{1}{2\pi\hbar} \int dp \, e^{\frac{i}{\hbar} p (q_{j+1} - q_j)}} - \frac{i\tau}{\hbar} \langle q_{j+1} | H | q_j \rangle$$

For $H = \frac{p^2}{2m} + V(q)$ we have:

$$\langle q_{j+1} | \frac{p^2}{2m} | q_j \rangle = \int dp' \, dp \, \underbrace{\langle q_{j+1} | p' \rangle}_{\frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i p' q_{j+1}}{\hbar}}} \underbrace{\langle p' | \frac{p^2}{2m} | p \rangle}_{\frac{p^2}{2m} \delta(p' - p)} \overbrace{\langle p | q_j \rangle}^{\frac{1}{\sqrt{2\pi\hbar}} e^{\frac{-i p q_j}{\hbar}}}$$

$$= \int dp \int dp' \, \frac{e^{\frac{i}{\hbar}(p' q_{j+1} - p q_j)}}{2\pi\hbar} \, \frac{p^2}{2m} \, \delta(p' - p)$$

$$= \int \frac{dp}{2\pi\hbar} \, e^{\frac{i}{\hbar} p (q_{j+1} - q_j)} \, \frac{p^2}{2m}$$

and $\langle q_{j+1} | V(q) | q_j \rangle \simeq V\left(\frac{q_{j+1} + q_j}{2} \right) \overbrace{\langle q_{j+1} | q_j \rangle}^{\delta(q_{j+1} - q_j)} =$

$$\int \frac{dp}{2\pi\hbar} \, e^{\frac{i}{\hbar} p (q_{j+1} - q_j)} \, V(\bar{q}_j) \quad , \quad \bar{q}_j = \frac{q_{j+1} + q_j}{2}$$

So $\langle q_{j+1} t_{j+1} | q_j t_j \rangle = \frac{1}{2\pi\hbar} \int dp_j \, e^{\frac{i}{\hbar}\left(p_j (q_{j+1} - q_j) - \tau H(p_j, \bar{q}_j) \right)}$

~~Begin~~

and

$$\langle q'' t'' | q' t' \rangle = \lim_{n \to \infty} \int \prod_{j=1}^{n} dq_j \prod_{j=0}^{n} \frac{dp_j}{2\pi\hbar} \exp \frac{i}{\hbar} \sum_{j=0}^{n} p_j (q_{j+1} - q_j) - \tau H(p_j, \bar{q}_j)$$

$= \int \frac{\mathcal{D}q \, \mathcal{D}p}{2\pi\hbar} \exp \frac{i}{\hbar} \int_{t'}^{t''} (p\dot{q} - H(p,q))$. The integral in $\mathcal{D}p$ is Gaussian since $H = \frac{p^2}{2m} + V(q)$ so it can be ~~perm~~ performed ~~by~~ and we get the Lagrangian appearing on $p\dot{q} - H$:

$$\langle q'' t'' | q' t' \rangle = N \int \mathcal{D}q \, \exp\left(\frac{i}{\hbar} \int_{t'}^{t''} L(q, \dot{q}) \, dt \right) .$$

2. Creation and annihilation operators

The hamiltonian for the free scalar field of mass m is

$$H = \frac{1}{2}\int \left((\partial_0 \varphi)^2 + |\nabla\varphi|^2 + m^2\varphi^2 \right) d\vec{x}$$

Using $\varphi(x) = \int \frac{d\vec{k}}{(2\pi)^3\, 2\omega(k)} \left(a(\vec{k})\, e^{-ikx} + a^{+}(\vec{k})\, e^{ikx} \right)$

we find, after some ~~dabt~~ calculations (and using some δ-functions which appear to kill one of the k-integrals — since we multiply $\partial\varphi \cdot \partial\varphi$ we'll have two):

$$H = \int \frac{\omega}{2} \left(\frac{a^{+}(\vec{k})\, a(\vec{k}) + a(\vec{k})\, a^{+}(\vec{k})}{(2\pi)^3\, 2\omega} \right) d\vec{k}$$

Define $N(k) = \frac{a^{+}(\vec{k})\, a(\vec{k})}{(2\pi)^3\, 2\omega}$, then, using $[a(\vec{k}), a^{+}(\vec{k}')] = (2\pi)^3\, 2\omega\, \delta(\vec{k} - \vec{k}')$:

$$\frac{1}{(2\pi)^3\, 2\omega}\left[a^{+}(\vec{k})\, a(\vec{k}) + a(\vec{k})\, a^{+}(\vec{k}) \right] = \frac{a^{+}(\vec{k})\, a(\vec{k})}{(2\pi)^3\, 2\omega} + \frac{[a(\vec{k}), a^{+}(\vec{k})]}{(2\pi)^3\, 2\omega} + \frac{a^{+}(\vec{k})\, a(\vec{k})}{(2\pi)^3\, 2\omega}$$

So $H = \int \omega \left(N(k) + \frac{1}{2}\,\delta(\underbrace{\vec{k} - \vec{k}}_{=0}) \right) d\vec{k}$

$$= \int \left(\omega N(k) + \frac{1}{2}\,\delta(0) \right) d\vec{k}$$

This is obviously not compatible with gravity; also, it's more delicate than this because of the Casimir effect; see [To]

To deal with the infinity contribution of $\delta(0)$ we remember that only differences of energy are physically significant, so we can drop the infinite (but constant!) δ term redefining:

$$:H: = \int \omega N(k)\, d\vec{k}$$ (notice that we then have $\langle 0 | H | 0 \rangle = 0$)

$: \, :$ means normal ordering which means that all annihilation operators should be written on the right; this guarantees that $\langle 0 | H | 0 \rangle = 0$. Normal ordering of any operator is defined by requiring that $\langle 0 | \mathcal{O} | 0 \rangle = 0$. (St 44)

Now it is easy to see that a state of the form $|k_1 \ldots k_n\rangle = a^\dagger(\vec{k}_1) \ldots a^\dagger(\vec{k}_n) |0\rangle$

This is an eigenstate of the (normal ordered) Hamiltonian with energy (= eigenvalue) $\omega_1 + \cdots + \omega_n$. For example

$$:H:\,|k_1\rangle = \int d\vec{k}\, \frac{a^\dagger(\vec{k})\,a(\vec{k})}{2(2\pi)^3}\,|k_1\rangle =$$

$$= \int \frac{d\vec{k}}{2(2\pi)^3}\, a^\dagger(\vec{k})\, \underbrace{a(k)\, a^\dagger(k_1)}\,|0\rangle$$

$$= \underbrace{[a(\vec{k}), a^\dagger(\vec{k}_1)]}_{(2\pi)^3\, 2\,\omega(\vec{k}_1)\,\delta(\vec{k}-\vec{k}_1)} + a^\dagger(\vec{k}_1)\,a(\vec{k})$$

$$= \int d\vec{k}\; \omega(\vec{k}_1)\, a^\dagger(\vec{k})\, \delta(\vec{k}-\vec{k}_1)\,|0\rangle + \int (\ )\, \underbrace{a(k)|0\rangle}_{=0}$$

$$= \omega(k_1)\, a^\dagger(k_1)\,|0\rangle = \omega(k_1)\,|k_1\rangle$$

So we ~~really~~ see that $a^\dagger(k)$ acting on the vacuum creates a particle with energy $\omega(\vec{k})$.

In the above calculation we used $a(k)|0\rangle = 0$. To see that this is really the case, consider

$$[N(k), N(k')] \propto a^\dagger(\vec{k})\,a(\vec{k})\,a^\dagger(\vec{k}')\,a(\vec{k}') - a^\dagger(\vec{k}')\,a(\vec{k}')\,a^\dagger(\vec{k})\,a(\vec{k})$$

$$= a^\dagger(\vec{k})\Big(\overbrace{[a(\vec{k}), a^\dagger(\vec{k}')]}^{\propto\delta(\vec{k}-\vec{k}')} + a^\dagger(\vec{k}')a(\vec{k})\Big)a(\vec{k}') -$$

$$- a^\dagger(\vec{k}')\Big(\underbrace{[a(\vec{k}')a^\dagger(\vec{k})]}_{\propto\delta(\vec{k}-\vec{k}')} + a^\dagger(\vec{k})a(\vec{k}')\Big)a(\vec{k})$$

$$= \left(a^\dagger(\vec{k})a(\vec{k}') - a^\dagger(\vec{k}')a(\vec{k})\right)\delta(k-\vec{k}')$$

$$+ a^\dagger(\vec{k})a^\dagger(\vec{k}')a(\vec{k})a(\vec{k}') - \underbrace{a^\dagger(\vec{k}')a^\dagger(\vec{k})}_{=a^\dagger(\vec{k})a^\dagger(\vec{k}')}\underbrace{a(\vec{k}')a(\vec{k})}_{=a(\vec{k})a(\vec{k}')}$$

since these operators commute

So, the last two terms cancel. The remaining term is zero; if $k\neq\vec{k}'$ then $\delta=0$; if $\vec{k}=\vec{k}'$ then what is inside the parenthesis is zero: $a^\dagger(\vec{k})a(\vec{k}) - a^\dagger(\vec{k})a(\vec{k}) = 0$

Therefore the operators $N(k)$, $N(k')$ commute, we can then diagonalize them simultaneously and use ~~its~~ its eigenvalues as basis: $N(k)\,|n(k)\rangle = n(k)\,|n(k)\rangle$.

A calculation shows:

$N(k)\,a^\dagger(k)\,|n(k)\rangle = (n(k)+1)\,(a^\dagger(k))\,|n(k)\rangle$

$\quad$ " $\quad a(\ )$ " $\quad (n(k)-1)\,(a(k))$ "

So $a^\dagger(k)$, $a(k)$ increase / decrease the eigenvalues of $N(k)$ by one. Now, from $H = \int \omega N(k)\, d\vec{k}$, and the fact that $|k\rangle = a^\dagger(k)|0\rangle$ is an eigenstate of H with energy ω, we see that N can be interpreted as a particle number operator, so we should have $a(\vec{k})|0\rangle = 0$.

3. Lorentz invariant measure

The construction of the Lorentz invariant measure can be found on RS p. 70ff (vol 2). Here we illustrate this as follows (which is sort of what RS does):

Consider the Lorentz invariant set $\{ k^2 > 0,\ k^0 > 0 \}$ and map it homeomorphically to $\mathbb{R}^3 \times \mathbb{R}_+$ by $(k_0, \vec{k}) \mapsto (\vec{k}, y)$, where $y = k^2$ (of course, for each k, y is just the mass). Now $\frac{\partial y}{\partial k_0} = 2k_0$ and then $dy = 2k_0\, dk_0 + (\quad)\, d\vec{k}^3$ and hence

$$dy \wedge d\vec{k}^3 = 2k_0\, dk_0 \wedge d\vec{k}^3$$

or $\frac{dy \wedge d\vec{k}^3}{2k_0} = dk$. Now dk is invariant because any Lorentz transformation satisfies $|\det \Lambda| = 1$. Moreover, since a Lorentz transformation takes $k^2 = m^2$ into itself, we have that

$$\frac{dy\, d\vec{k}^3\, \delta(k^2 - m^2)}{2k_0} = dk\, \delta(k^2 - m^2)$$

is still Lorentz invariant (since the right hand side is), and so is

$$\frac{d\vec{k}}{2k_0} = \int_{\mathbb{R}_+} \frac{dy\, d\vec{k}}{2k_0}\, \delta(k^2 - m^2)$$

4. The free hamiltonian

$$\varphi(x) = \int \left(a(\vec k)\, e^{-ikx} + a^\dagger(\vec k)\, e^{ikx} \right) \frac{d\vec k}{(2\pi)^3 2\omega}$$

the hamiltonian density is (See 4a)

$$\mathcal{H} = \frac{1}{2}\dot\varphi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2$$

$$= \frac{1}{2}\iint \frac{d\vec k}{(2\pi)^3 2\omega}\frac{d\vec k'}{(2\pi)^3 2\omega'} \left(-i\omega\, a(\vec k)\, e^{-ikx} + i\omega\, a^\dagger(\vec k)\, e^{ikx} \right) \times$$

$$\left(-i\omega'\, a(\vec k')\, e^{-ik'x} + i\omega'\, a^\dagger(\vec k')\, e^{ik'x} \right) +$$

$$\left(+i\vec k\, a(\vec k)\, e^{-ikx} - i\vec k\, a^\dagger(\vec k)\, e^{ikx} \right)\left(i\vec k'\, a(\vec k')\, e^{-ik'x} - i\vec k'\, a^\dagger(\vec k')\, e^{ik'x} \right)$$

$$+ m^2 \left(a(\vec k)\, e^{-ikx} + a^\dagger(\vec k)\, e^{ikx} \right)\left(a(\vec k')\, e^{-ik'x} + a^\dagger(\vec k')\, e^{ik'x} \right)$$

$$= \frac{1}{2}\iint \frac{d\vec k\, d\vec k'}{(2\pi)^3 2\omega\, (2\pi)^3 2\omega'} \Big\{ a(\vec k)\, a(\vec k') \left(-\omega\omega' - \vec k\cdot\vec k' + m^2 \right) e^{-i(k+k')x} +$$

$$a(\vec k)\, a^\dagger(\vec k') \left(\omega\omega' + \vec k\cdot\vec k' + m^2 \right) e^{-i(k-k')x} +$$

$$a^\dagger(\vec k)\, a(\vec k') \left(\omega\omega' + \vec k\cdot\vec k' + m^2 \right) e^{i(k-k')x} +$$

$$a^\dagger(\vec k)\, a^\dagger(\vec k') \left(-\omega\omega' - \vec k\cdot\vec k' + m^2 \right) e^{i(k+k')x} \Big\}$$

Since $H = \int \mathcal{H} \, d\vec{x}$, we obtain:

$$H = \frac{1}{2} \iint \frac{d\vec{k}}{2\omega} \frac{d\vec{k}'}{2\omega'} \Big\{ (-\omega\omega' - \vec{k}\cdot\vec{k}' + m^2) \Big[a(\vec{k})\, a(\vec{k}')\, e^{-i(\omega+\omega')t} + a^{\dagger}(\vec{k})\, a^{\dagger}(\vec{k}')\, e^{i(\omega+\omega')t} \Big] \Big[\underbrace{\int \frac{d\vec{x}}{(2\pi)^3} e^{i(\vec{k}+\vec{k}')\cdot\vec{x}}}_{=\delta(\vec{k}+\vec{k}')} + \underbrace{\int \frac{d\vec{x}}{(2\pi)^3} e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}}}_{=\delta(\vec{k}+\vec{k}')} \Big]$$

$$+ (\omega\omega' + \vec{k}\cdot\vec{k}' + m^2) \Big[a(\vec{k})\, a^{\dagger}(\vec{k}')\, e^{-i(\omega-\omega')t} + a^{\dagger}(\vec{k})\, a(\vec{k}')\, e^{i(\omega-\omega')t} \Big]$$

$$\Big[\underbrace{\int \frac{d\vec{x}}{(2\pi)^3} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}}_{=\delta(\vec{k}-\vec{k}')} + \underbrace{\int \frac{d\vec{x}}{(2\pi)^3} e^{-i(\vec{k}-\vec{k})\cdot\vec{x}}}_{=\delta(\vec{k}-\vec{k})} \Big] \Big\}$$

Using the deltas (notice that $\delta + \delta = 2\delta$, so 2 cancels $\frac{1}{2}$):

$$= \int \frac{d\vec{k}}{(2\omega)^2} \Big\{ (-\omega^2 + \vec{k}^2 + m^2) \Big[a(\vec{k})\, a(-\vec{k})\, e^{-i2\omega t} + a^{\dagger}(\vec{k})\, a^{\dagger}(-\vec{k})\, e^{i2\omega t} \Big]$$

$$+ (\omega^2 + \vec{k}^2 + m^2) \Big(a(\vec{k})\, a^{\dagger}(\vec{k}) + a^{\dagger}(\vec{k})\, a(\vec{k}) \Big) \Big]$$

Now $k^2 = m^2 = \omega^2 - \vec{k}^2$ so $-\omega^2 + \vec{k}^2 + m^2 = 0$ what kills the first [ ]. And $\omega^2 = \vec{k}^2 + m^2$ so $\omega^2 + \vec{k}^2 + m^2 = 2\omega^2$

$$H = \frac{1}{2} \int \frac{d\vec{k}}{2\omega} \, 2\omega \left(a(\vec{k}) a^\dagger(\vec{k}) + a^\dagger(\vec{k}) a(\vec{k}) \right)$$

$$H = \cancel{\frac{1}{2}} \int \frac{d\vec{k}}{2} \left(a(\vec{k}) a^\dagger(\vec{k}) + a^\dagger(\vec{k}) a(\vec{k}) \right)$$

5. The functional derivative

The directional derivative of a functional $A(\varphi)$ on $\mathcal{D}'$ (= distributions) in the direction of ψ is defined by

$$(D_\psi A)(\varphi) \equiv (DA)(\varphi)(\psi) = \lim_{\varepsilon \to 0} \frac{A(\varphi + \varepsilon\psi) - A(\varphi)}{\varepsilon}$$

where $\psi \in \mathcal{D}'$ and the derivative may exist for a particular A pointwise, w.r.t. a measure $d\mu(\varphi)$, in L^p etc.

The special case $\psi = \delta_x$ arises often and it is given a special notation. $D_{\delta_x} A(\varphi) = \frac{\delta A(\varphi)}{\delta \varphi(x)}$ (see JG p. 167). However, ~~phys~~ physicists write the following all the time. $\frac{\delta \varphi(y)}{\delta \varphi(x)} = \delta(x-y)$. If $\varphi \in \mathcal{D}'$, its pointwise value doesn't make sense in general, and if φ is a function then this doesn't make sense.

We interpret this in the following way. For a fixed point y consider a family of Gaussian functions f_σ which converges in $\mathcal{D}'$ to δ_y: $f_\sigma \to \delta_y$ when $\sigma \to 0$. We can modify f_σ to have compact support, or take another family of approximating functions or even suppose we are working with tempered distributions instead. In all these cases we have then that f_σ define a functional on $\mathcal{D}'$ by $A(\varphi) := \varphi(f_\sigma)$. So

$$\frac{\delta A}{\delta\varphi(x)} = \lim_{\varepsilon\to 0} \frac{A(\varphi+\varepsilon\delta_x) - A(\varphi)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{(\varphi+\varepsilon\delta_x)(f_\sigma) - \varphi(f_\sigma)}{\varepsilon}$$

$$= \delta_x(f_\sigma) = f_\sigma(x)$$

And now this converges in $\mathcal{D}'$ to δ_y. So in the physicist expression $\int \frac{\delta\varphi(x)}{\delta\varphi(y)} g(x)\,dx = \int \delta(x-y) g(x)\,dx = g(y)$ it is understood that f_σ is identified with the distribution given by integration against it and it is also understood a limit:

$$\frac{\delta A}{\delta\varphi(x)}(g) = f_\sigma(x)(g) = \int f_\sigma(x) g(x)\,dx \longrightarrow \delta_y(g) = g(y).$$

Of course, given a function $\varphi(y)$ we can make it into a distribution by integration and then the same reasoning applies.

Remarks

~~Last time~~

~~Amplitudes in QM~~

$$\cancel{\langle q'' t'' | T(Q(t_1)\cdots Q(t_n)) | q' t' \rangle = N \int \mathcal{D}q \; q(t_1)\cdots q(t_n)\, e^{\frac{iS}{\hbar}}}$$

~~Caution! physicists don't write $T(\ )$~~

Quantization of the free field

solve $\Box \varphi + m^2 \varphi = 0 \Rightarrow \varphi(t,\vec{x}) = \int \left(a(\vec{k}) e^{-ikx} + a^*(\vec{k}) e^{ikx} \right) \frac{d\vec{k}}{(2\pi)^3 2\omega}$

impose C.R. $[\varphi(t,\vec{x}), \pi(t,\vec{x}')] = i\delta(\vec{x}-\vec{x}')$ and get

$\pi(t,\vec{x}) = \dot{\varphi}(t,\vec{x})$

$$a^+(\vec{k}) = -i \int e^{-ikx} \overleftrightarrow{\partial_0} \varphi(x)\, d\vec{x} \;, \quad a(\vec{k}) = i \int e^{ikx} \overleftrightarrow{\partial_0} \varphi(x)\, d\vec{x}$$

$$[a(\vec{k}), a^+(\vec{k}')] = (2\pi)^3\, 2\omega(\vec{k})\, \delta(\vec{k}-\vec{k}')$$

~~[illegible]~~

~~Then~~ Physicists would construct the Fock space by taking the tensor product over all wavevectors $\vec{k}$ (SH (?))

$\mathcal{F} = \bigotimes_{\vec{k}} H_k$, where $H_k = \mathrm{span}\{ a^+(\vec{k})|0\rangle \}$. But in practice only finite tensor products are used and in the scattering process whose probability we want

R1

to compute we know a priori the values $\vec{k}_1, \dots, \vec{k}_n$ which may be involved. Then we take (GJ 103)

$$\mathcal{F} = \bigotimes_{\vec{k} \in \left(\frac{\pi}{L}\mathbb{Z}_+\right)^3} \mathcal{H}_k \qquad \text{(see the book for more details)}$$

Another way of thinking of $\mathcal{F}$ is the following. We consider $\mathcal{H} = \operatorname{span}\{ a(\vec{k})|0\rangle \mid \vec{k} \in \mathbb{R}^3\}$ and put (as usual in math)

$$\mathcal{F} = \mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \oplus \cdots$$

and now we extend the action of $a(\vec{k})$ and $a^\dagger(\vec{k})$ to each $\mathcal{H}^n = \underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n \text{ times}}$ by

$$a^\dagger_{(n)}(\vec{k})|k_1 \dots k_n\rangle = a^\dagger_{(n)}(\vec{k})|k_1\rangle \otimes \cdots \otimes |k_n\rangle := |k_1\rangle \otimes \cdots \otimes |k_n\rangle \otimes a^\dagger(\vec{k})|0\rangle$$

$= |k_1 \dots k_n \vec{k}\rangle$.

$$a_{(n)}(\tilde{\vec{k}})|k_1 \dots k_n\rangle = \left(a(\tilde{\vec{k}}) \otimes id \otimes \cdots \otimes id + id \otimes a(\tilde{\vec{k}}) \otimes \cdots id + id \otimes \cdots \otimes a(\vec{k})\right)|k_1\rangle \otimes \cdots \otimes |k_n\rangle.$$

Of course, if $\vec{k} \neq k_i \ \forall i$ then this equals zero [recall $[a(\vec{k}), a^\dagger(\tilde{\vec{k}})] = 0$ if $\vec{k} \neq \tilde{\vec{k}}$ so $a(\vec{k})|k_i\rangle = a(\tilde{\vec{k}})a^\dagger(\vec{k}_i)|0\rangle = a^\dagger(\vec{k}_i)a(\vec{k})|0\rangle = 0$]

and if $\vec{k}_j = \vec{k}$ for some j:

$$= 0 + \cdots |k_1\rangle \otimes \cdots \otimes \underbrace{a(\vec{k})|\vec{k}_j\rangle}_{=|0\rangle} \otimes \cdots \otimes |k_n\rangle$$

$$= |\vec{k}_1\rangle \otimes \cdots \otimes |0\rangle \otimes \cdots |\vec{k}_n\rangle \equiv |k_1 \hat{k}_j k_n\rangle .$$

So when we write, e.g. $a^\dagger(\vec{k}_1)a^\dagger(\vec{k}_2)|0\rangle$ it's implicit that, since $a(\vec{k}_2)|0\rangle \in \mathcal{H}$, $a^\dagger(\vec{k}_1)$ actually means $a^\dagger_{(2)}(\vec{k}_1)$ etc. (see St 42)

So this corresponds exactly to what physicists do when they write, e.g., $a(\vec{k}_1)|k_1 k_2\rangle = |k_2\rangle$ Indeed, they notation for the "continuum" Fock space is "listing the momentum of each non-zero excitation" (St. 41)

Of course, we are not taking the symmetrized Fock space, if we ~~do~~ do so the way $a_{(n)}, a^\dagger_{(n)}$ are defined above will be different. Indeed, ~~to~~ see quantization ~~of~~ on RS vol 2.

R3

Another important remark concerns the notation $\langle\psi|A|\varphi\rangle$.

Rewriting in a mathematical notation this means:

$$\langle\psi|(A|\varphi\rangle) = \langle\psi, A\varphi\rangle$$

but

$$(\langle\psi|A)|\varphi\rangle = \langle A^\dagger\psi, \varphi\rangle$$

So when we write $\langle\psi|A|\varphi\rangle$ we dont say in which side A acts but when it acts on the left we really need to take the adjoint. To see that it is really this what physics books mean, consider the example

$$|i\rangle = a^\dagger(\vec{k}_1)|0\rangle, \quad |f\rangle = a^\dagger(k_2)|0\rangle \quad \text{so}$$

$\langle f|i\rangle = \langle 0|a(\vec{k}_2)\, a^\dagger(\vec{k}_1)|0\rangle$. Now this means that acting on the right $a(\vec{k}_2)$ acts as annihilation operator, exactly as $\langle f|i\rangle = \langle a^\dagger(\vec{k}_2)0, a^\dagger(\vec{k}_1)0\rangle = \langle 0, a(\vec{k}_2)\, a^\dagger(\vec{k}_1)0\rangle$. But on the left clearly $\langle 0|a(\vec{k}_2)$ does not mean act as

$a(\vec{k}_2)|0\rangle$ and then take the result and inner product it with $a^\dagger(\vec{k}_1)|0\rangle$ since $a(\vec{k}_2)|0\rangle = 0$. $\langle 0| a(\vec{k}_2)$ actually means the ~~~~ functional corresponding to $a^\dagger(\vec{k}_2)|0\rangle$; so we have, in physicist notation $\left(a^\dagger(\vec{k}_2)|0\rangle\right)^\dagger = \langle 0| a(\vec{k}_2)$.

References

[Ry] Ryder, Quantum Field Theory

[St] Sterman, An introduction to QFT

[Sre] Srednicki, QFT, online notes on his webpage.

[RS] Reed and Simon, Methods of modern Mathematical Physics, vol 1 & 2 (most references are in fact to pages in vol 2)

[GJ] Glimm and Jaffe, Quantum physics, a functional integral point of view.

[Ze] Zee, QFT in a nutshell

[To] Tong, QFT. Textbook in his webpage, Cambridge University