REAL ANALYSIS, HW 4

VANDERBILT UNIVERSITY

supp	Support of a function or a measure
X	Locally compact (topological) space
K	Compact set in X
E	Locally convex (topological vector) space
$\mathscr{C}(X;E)$	Space of continuous functions from X to E endowed with the uniform topology
$\mathscr{C}_{c.o.}(X;E)$	Space of continuous functions from X to E endowed with the compact-open topology
$\mathscr{C}_c(X;E)$	Space of continuous functions from X to E with compact support endowed with the compact-open topology
$\mathscr{C}(K;E)$	Space of continuous functions from K to E endowed with the topology inherited from $\mathscr{C}(X,E)$
$\mathscr{K}(X;E)$	Space of continuous functions from X to E with compact support endowed with the inductive limit of locally convex topologies
$\mathscr{K}(X,A;E)$	Elements $f \in \mathscr{K}(X;E)$ such that $\text{supp}(f) \subseteq A$
$\mathscr{K}(X,K;E)$	Elements $f \in \mathscr{K}(X;E)$ such that $\text{supp}(f) \subseteq K$ endowed with the topology of compact convergence
$\mathscr{K}_+(X;\mathbb{R})$	Elements $f \in \mathscr{K}(X;\mathbb{R})$ such that $f \geq 0$
$\mathscr{K}(X)$	$\mathscr{K}(X;\mathbb{C})$ or $\mathscr{K}(X;\mathbb{R})$, with $\mathbb{C}$ or $\mathbb{R}$ understood from the context
$\mathscr{M}(X;\mathbb{C})$	Space of measures on X
$\mathscr{M}(X;\mathbb{R})$	Space of real measures on X
$\mathscr{M}_+(X;\mathbb{R})$	Space of positive measures on X
$\mathscr{I}_+(X;\mathbb{R})$	Space of positive (non-negative) lower semi-continuous functions on X
$\mu^*(f)$	Upper integral of f (with respect to the positive measure μ), also denoted $\int^* f\, d\mu$
χ_A	Characteristic function of the set A
$\mu^*(A)$	Outer measure of A (with respect to the positive measure μ)
$N_p(f)$	$(\lvert\mu\rvert^*(\lvert f\rvert^p))^{\frac{1}{p}}$, $1 \leq p < \infty$
$\mathscr{F}^p(X)$	Maps f from X to $\mathbb{C}$ or $\mathbb{R}$ such that $N_p(f) < \infty$, with topology given by the semi-norm N_p. Depending on the context, $\mathscr{F}^p(X)$ can denote maps defined a.e. such that $N_p(f) < \infty$, and also taking values in $\overline{\mathbb{R}}$
$\mathscr{L}^p(X)$	Closure of $\mathscr{K}(X)$ in $\mathscr{F}^p(X)$
$\mathscr{L}^p_{loc}(X)$	Functions $f : X \to \mathbb{C}$ such that $f\chi_K \in \mathscr{L}^p_{loc}(X)$ for every compact set $K \subseteq X$.
$L^p(X)$	Hausdorff space associated with $\mathscr{L}^p(X)$
$f \sim g$	Equivalence relation $f(x) = g(x)$ a.e.
$\widetilde{f}$	Equivalence class of f given by the equivalence relation $\sim$
$\mathscr{E}_F(\Phi)$	Set of Φ-step functions with values in F, where Φ is a Boolean ring and $F = \mathbb{R}$ or $\mathbb{C}$.

Recall that we also call the compact-open topology the topology of compact convergence. Unless stated otherwise, the ordering in the function spaces and spaces of measures is as defined in class and denoted $\leq$, when such relation is well-defined. Recall that by a set of zero measure we mean a set of zero outer measure. The topology on $\mathscr{F}^p(X)$ is called the topology of convergence of mean of order p, the L^p-topology, or yet the topology of convergence in L^p. Elements in $\mathscr{L}^p(X)$ are called p-integrable. This terminology is extended to functions defined a.e. and taking values in $\overline{\mathbb{R}}$ as done in class.

Below, the measure μ on $\mathbb{R}^n$ is always the Lebesgue measure.

Question 1. Recall that in class we defined the Hardy-Littlewood maximal operator M as follows. Given $f \in \mathscr{L}^1_{loc}(\mathbb{R}^n)$, set

$$(Mf)(x) = \sup_{r>0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f| \, d\mu.$$

Show that Mf is lower semi-continuous and that the set $A_\lambda = \{x \in \mathbb{R}^n \,|\, |(Mf)(x)| > \lambda\}$ is open for each $\lambda > 0$.

Question 2. Continuing from question 1, and assuming now that f is integrable, show that there exists a compact set $K \subseteq A_\lambda$ such that $2\mu(K) \geq \mu(A_\lambda)$, and that for each $x \in K$ there exists a ball $B_\rho(x)$, where ρ depends on x, such that

$$\frac{1}{\mu(B_\rho(x))} \int_{B_\rho(x)} |f| \, d\mu > \lambda.$$

Show that there exist finitely many $\{B_\rho(x_i)\}_{i=1}^N$ of the these balls that are pair-wise disjoint and such that $\{B_{3\rho}(x_i)\}_{i=1}^N$ covers K.

Remark. Question 2 is an intermediate step to prove the inequality of question 3, which is trivial if f is not integrable. So we assume $f \in \mathscr{L}^1(\mathbb{R}^n)$. (To put this in context, recall that problem 3 was left as an exercise in the proof of Lebesgue's differentiation theorem.)

Question 3. Continuing from question 2, conclude that

$$\mu(A_\lambda) \leq \frac{2 \cdot 3^n}{\lambda} N_1(f).$$

Question 4. Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map, where U is open, and let $A \subset U$ be a negligible set. Show that $f(A)$ is negligible.

Question 5. Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map, where U is open, and let $A \subseteq U$ be measurable set. Show that $f(A)$ is measurable.