REAL ANALYSIS, HW 4

VANDERBILT UNIVERSITY

supp	Support of a function or a measure
X	Locally compact (topological) space
K	Compact set in X
E	Locally convex (topological vector) space
$\mathscr{C}(X;E)$	Space of continuous functions from X to E endowed with the uniform topology
$\mathscr{C}_{c.o.}(X;E)$	Space of continuous functions from X to E endowed with the compact-open topology
$\mathscr{C}_c(X;E)$	Space of continuous functions from X to E with compact support endowed with the compact-open topology
$\mathscr{C}(K;E)$	Space of continuous functions from K to E endowed with the topology inherited from $\mathscr{C}(X,E)$
$\mathscr{K}(X;E)$	Space of continuous functions from X to E with compact support endowed with the inductive limit of locally convex topologies
$\mathscr{K}(X,A;E)$	Elements $f \in \mathscr{K}(X;E)$ such that $\text{supp}(f) \subseteq A$
$\mathscr{K}(X,K;E)$	Elements $f \in \mathscr{K}(X;E)$ such that $\text{supp}(f) \subseteq K$ endowed with the topology of compact convergence
$\mathscr{K}_+(X;\mathbb{R})$	Elements $f \in \mathscr{K}(X;\mathbb{R})$ such that $f \geq 0$
$\mathscr{K}(X)$	$\mathscr{K}(X;\mathbb{C})$ or $\mathscr{K}(X;\mathbb{R})$, with $\mathbb{C}$ or $\mathbb{R}$ understood from the context
$\mathscr{M}(X;\mathbb{C})$	Space of measures on X
$\mathscr{M}(X;\mathbb{R})$	Space of real measures on X
$\mathscr{M}_+(X;\mathbb{R})$	Space of positive measures on X
$\mathscr{I}_+(X;\mathbb{R})$	Space of positive (non-negative) lower semi-continuous functions on X
$\mu^*(f)$	Upper integral of f (with respect to the positive measure μ), also denoted $\int^* f\, d\mu$
χ_A	Characteristic function of the set A
$\mu^*(A)$	Outer measure of A (with respect to the positive measure μ)
$N_p(f)$	$(\lvert\mu\rvert^*(\lvert f\rvert^p))^{\frac{1}{p}}$, $1 \leq p < \infty$
$\mathscr{F}^p(X)$	Maps f from X to $\mathbb{C}$ or $\mathbb{R}$ such that $N_p(f) < \infty$, with topology given by the semi-norm N_p. Depending on the context, $\mathscr{F}^p(X)$ can denote maps defined a.e. such that $N_p(f) < \infty$, and also taking values in $\overline{\mathbb{R}}$
$\mathscr{L}^p(X)$	Closure of $\mathscr{K}(X)$ in $\mathscr{F}^p(X)$
$\mathscr{L}^p_{loc}(X)$	Functions $f : X \to \mathbb{C}$ such that $f\chi_K \in \mathscr{L}^p_{loc}(X)$ for every compact set $K \subseteq X$.
$L^p(X)$	Hausdorff space associated with $\mathscr{L}^p(X)$
$f \sim g$	Equivalence relation $f(x) = g(x)$ a.e.
$\widetilde{f}$	Equivalence class of f given by the equivalence relation $\sim$
$\mathscr{E}_F(\Phi)$	Set of Φ-step functions with values in F, where Φ is a Boolean ring and $F = \mathbb{R}$ or $\mathbb{C}$.

Recall that we also call the compact-open topology the topology of compact convergence. Unless stated otherwise, the ordering in the function spaces and spaces of measures is as defined in class and denoted $\leq$, when such relation is well-defined. Recall that by a set of zero measure we mean a set of zero outer measure. The topology on $\mathscr{F}^p(X)$ is called the topology of convergence of mean of order p, the L^p-topology, or yet the topology of convergence in L^p. Elements in $\mathscr{L}^p(X)$ are called p-integrable. This terminology is extended to functions defined a.e. and taking values in $\overline{\mathbb{R}}$ as done in class.

Below, the measure μ on $\mathbb{R}^n$ is always the Lebesgue measure.

Question 1. Recall that in class we defined the Hardy-Littlewood maximal operator M as follows. Given $f \in \mathscr{L}^1_{loc}(\mathbb{R}^n)$, set

$$(Mf)(x) = \sup_{r>0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f| \, d\mu.$$

Show that Mf is lower semi-continuous and that the set $A_\lambda = \{x \in \mathbb{R}^n \,|\, (Mf)(x)| > \lambda\}$ is open for each $\lambda > 0$.

Solution. Let

$$f_r(x) = \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f| \, d\mu.$$

We claim that f_r is a continuous function.

Fix $0 < r < R$ and let $m = \mu(B_r(0))$ (which of course is equatl to $\mu(B_r(z))$ for any $z \in \mathbb{R}^n$). Since $f\chi_{B_R(0)}$ is integrable, we can find a continuous compactly supported g such that

$$\int |\frac{f}{m}\chi_{B_R(0)} - g| \leq \varepsilon,$$

for a given $\varepsilon > 0$. Notice that this implies

$$\begin{aligned}
\int_{B_R(0)^c} |g| &= \int_{B_R(0)^c} |\frac{f}{m}\chi_{B_R(0)} - g| \\
&= \int |\frac{f}{m}\chi_{B_R(0)} - g|\chi_{B_R(0)^c} \\
&\leq \int |\frac{f}{m}\chi_{B_R(0)} - g| \leq \varepsilon,
\end{aligned}$$

and thus,

$$\begin{aligned}
\int_{B_R(0)} |\frac{f}{m} - g| &= \int |\frac{f}{m} - g|\chi_{B_R(0)} \\
&\leq \int |\frac{f}{m}\chi_{B_R(0)} - g| + \int |g - g\chi_{B_R(0)}| \\
&= \int |\frac{f}{m}\chi_{B_R(0)} - g| + \int_{B_R(0)^c} |g| \leq 2\varepsilon.
\end{aligned}$$

Consider x and y such that $B_r(x) \cup B_r(y) \subset B_R(0)$. Then

$$\begin{aligned}
|f_r(x) - f_r(y)| &= \left| \int (\frac{f}{m}\chi_{B_r(x)} - \frac{f}{m}\chi_{B_r(y)}) \right| \\
&\leq \int |\frac{f}{m}\chi_{B_r(x)} - g\chi_{B_r(x)}| + \int |g\chi_{B_r(x)} - g\chi_{B_r(y)}| + \int |g\chi_{B_r(y)} - \frac{f}{m}\chi_{B_r(y)}| \\
&= \int |\frac{f}{m} - g|\chi_{B_r(x)} + \int |g||\chi_{B_r(x)} - \chi_{B_r(y)}| + \int |g \ - \frac{f}{m}|\chi_{B_r(y)} \\
&\leq \int |\frac{f}{m} - g|\chi_{B_R(0)} + \int |g||\chi_{B_r(x)} - \chi_{B_r(y)}| + \int |g \ - \frac{f}{m}|\chi_{B_R(0)} \\
&= \int_{B_R(0)} |\frac{f}{m} - g| + \int_B |g| + \int_{B_R(0)} |g \ - \frac{f}{m}|,
\end{aligned}$$

where $B = (B_r(x) \cup B_r(y)) \backslash (B_r(x) \cap B_r(y))$. We have seen that the first and third integrals are $\leq 2\varepsilon$. Applying the mean value inequality to the second integral we have

$$\int_B |g| \leq \sup_{x \in \mathbb{R}^n} |g(x)|\mu(B),$$

which can be made as small as we want by taking x and y sufficiently close, and the claim is proven. So Mf is the supremum over a family of continuous functions thus it is lower-semicontinuous. This implies that A_λ is open.

Question 2. Continuing from question 1, and assuming now that f is integrable, show that there exists a compact set $K \subseteq A_\lambda$ such that $2\mu(K) \geq \mu(A_\lambda)$, and that for each $x \in K$ there exists a ball $B_\rho(x)$, where ρ depends on x, such that

$$\frac{1}{\mu(B_\rho(x))} \int_{B_\rho(x)} |f| \, d\mu > \lambda.$$

Show that there exist finitely many $\{B_\rho(x_i)\}_{i=1}^N$ of the these balls that are pair-wise disjoint and such that $\{B_{3\rho}(x_i)\}_{i=1}^N$ covers K.

Remark. Question 2 is an intermediate step to prove the inequality of question 3, which is trivial if f is not integrable. So we assume $f \in \mathscr{L}^1(\mathbb{R}^n)$. (To put this in context, recall that problem 3 was left as an exercise in the proof of Lebesgue's differentiation theorem.)

Solution. Let us start with the following claim: Assume that $L \subset \mathbb{R}^n$ is compact and let $\{B_{r_\alpha}(x_\alpha)\}_{\alpha \in A}$ be a covering of L by open balls. Then there exits a finite sub-collection $\{B_{r_i}(x_i)\}_{i=1}^N$ of pair-wise disjoint balls such that $\{B_{3r_i}(x_i)\}_{i=1}^N$ covers L.

We can take the original cover finite to begin with. Let $B_{r_1}(x_1)$ be the (not necessarily unique) ball with the largest radius. Let $B_{r_2}(x_2)$ be the ball with largest radius among those that are disjoint from $B_{r_1}(x_1)$ (again, $B_{r_2}(x_2)$ does not have to be unique). Continue inductively until this process ends in N steps. It suffices now to show that $B_{r_\alpha}(x_\alpha) \subseteq \cup_{i=1}^N B_{3r_i}(x_i)$ for any α. Fix α and notice that if $\alpha = i$ for some i in $\{1, \ldots, N\}$ then there is nothing to be showed. If $\alpha \neq i$ for all $i = 1, \ldots, N$, let i_* be the first index i in $\{1, \ldots, N\}$ such that $B_{r_{i_*}}(x_{i_*}) \cap B_{r_\alpha}(x_\alpha) \neq \varnothing$, which must exist since otherwise the process would not have stopped. Then $r_{i_*} \geq r_\alpha$ because otherwise $B_{r_{i_*}}(x_{i_*})$ would have been incorrectly selected. Invoking the triangle inequality we see that $B_{r_\alpha}(x_\alpha) \subseteq B_{3r_{i_*}}(x_{i_*})$

Remark. The above claim (and variants of it) is sometimes called *Vitali's covering theorem.*

Next, consider a compact set $L \subseteq A_\lambda$. The definition of Mf implies that for each $x \in L$, there exists a r_x such that $\lambda < f_r(x)$. The corresponding balls $B_{r_x}(x)$ (in the definition of f_r) cover L, thus by the previous claim we can find a finite collection $\{B_{r_i}(x_i)\}_{i=1}^N$ of pair-wise disjoint balls such that $\{B_{3r_i}(x_i)\}_{i=1}^N$ covers L. Thus

$$\mu(L) \leq \sum_{i=1}^{N} \mu(B_{3r_i}(x_i)) = 3^n \sum_{i=1}^{N} \mu(B_{r_i}(x_i)) \leq \frac{1}{\lambda} \int_L |f| \, d\mu.$$

Now, since A_λ is measurable, it can be written, up to a negligible set, as the disjoint union of a family of compact sets. By the foregoing, $\mu(A_\lambda) < \infty$ (recall that $f \in \mathscr{L}^1(\mathbb{R}^n)$), and thus the existence of the desired compact set K is immediate.

Question 3. Continuing from question 2, conclude that

$$\mu(A_\lambda) \leq \frac{2 \cdot 3^n}{\lambda} N_1(f).$$

Solution. This follows from the construction of problem 2.

Question 4. Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map, where U is open, and let $A \subset U$ be a negligible set. Show that $f(A)$ is negligible.

Solution. It all comes down to show that if R is a cube contained in U, then $f(R \cap A)$ is negligible. This is immediate if $\overline{R} \subset U$ in light of the similar statement proven in class for Lipschitz functions, since f is then Lipschitz on $\overline{R}$. In the general case, we can write R as a countable union of concentric closed cubes, hence the result.

Question 5. Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map, where U is open, and let $A \subseteq U$ be measurable set. Show that $f(A)$ is measurable.

Solution. Since A is measurable, we can write $A = N \cup \cup_n K_n$, where N is negligible and the K_n's are compact. $f(N)$ is negligible by question 4, hence the result as f is continuous.