REAL ANALYSIS, HW 3

VANDERBILT UNIVERSITY

supp	Support of a function or a measure
X	Locally compact (topological) space
K	Compact set in X
E	Locally convex (topological vector) space
$\mathscr{C}(X;E)$	Space of continuous functions from X to E endowed with the uniform topology
$\mathscr{C}_{c.o.}(X;E)$	Space of continuous functions from X to E endowed with the compact-open topology
$\mathscr{C}_c(X;E)$	Space of continuous functions from X to E with compact support endowed with the compact-open topology
$\mathscr{C}(K;E)$	Space of continuous functions from K to E endowed with the topology inherited from $\mathscr{C}(X,E)$
$\mathscr{K}(X;E)$	Space of continuous functions from X to E with compact support endowed with the inductive limit of locally convex topologies
$\mathscr{K}(X,A;E)$	Elements $f \in \mathscr{K}(X;E)$ such that $\text{supp}(f) \subseteq A$
$\mathscr{K}(X,K;E)$	Elements $f \in \mathscr{K}(X;E)$ such that $\text{supp}(f) \subseteq K$ endowed with the topology of compact convergence
$\mathscr{K}_+(X;\mathbb{R})$	Elements $f \in \mathscr{K}(X;\mathbb{R})$ such that $f \geq 0$
$\mathscr{K}(X)$	$\mathscr{K}(X;\mathbb{C})$ or $\mathscr{K}(X;\mathbb{R})$, with $\mathbb{C}$ or $\mathbb{R}$ understood from the context
$\mathscr{M}(X;\mathbb{C})$	Space of measures on X
$\mathscr{M}(X;\mathbb{R})$	Space of real measures on X
$\mathscr{M}_+(X;\mathbb{R})$	Space of positive measures on X
$\mathscr{I}_+(X;\mathbb{R})$	Space of positive (non-negative) lower semi-continuous functions on X
$\mu^*(f)$	Upper integral of f (with respect to the positive measure μ), also denoted $\int^* f\, d\mu$
χ_A	Characteristic function of the set A
$\mu^*(A)$	Outer measure of A (with respect to the positive measure μ)
$N_p(f)$	$(\lvert\mu\rvert^*(\lvert f\rvert^p))^{\frac{1}{p}}$, $1 \leq p < \infty$
$\mathscr{F}^p(X)$	Maps f from X to $\mathbb{C}$ or $\mathbb{R}$ such that $N_p(f) < \infty$, with topology given by the semi-norm N_p. Depending on the context, $\mathscr{F}^p(X)$ can denote maps defined a.e. such that $N_p(f) < \infty$, and also taking values in $\overline{\mathbb{R}}$
$\mathscr{L}^p(X)$	Closure of $\mathscr{K}(X)$ in $\mathscr{F}^p(X)$
$L^p(X)$	Hausdorff space associated with $\mathscr{L}^p(X)$
$f \sim g$	Equivalence relation $f(x) = g(x)$ a.e.
$\widetilde{f}$	Equivalence class of f given by the equivalence relation $\sim$
$\mathscr{E}_F(\Phi)$	Set of Φ-step functions with values in F, where Φ is a Boolean ring and $F = \mathbb{R}$ or $\mathbb{C}$.

Recall that we also call the compact-open topology the topology of compact convergence. Unless stated otherwise, the ordering in the function spaces and spaces of measures is as defined in class and denoted $\leq$, when such relation is well-defined. Recall that by a set of zero measure we mean a set of zero outer measure. The topology on $\mathscr{F}^p(X)$ is called the topology of convergence of mean of order p, the L^p-topology, or yet the topology of convergence in L^p. Elements in $\mathscr{L}^p(X)$ are called p-integrable. This terminology is extended to functions defined a.e. and taking values in $\overline{\mathbb{R}}$ as done in class.

X will always denote a locally compact σ-compact space.

Question 1. Let μ and ν be positive measures on X. Prove that ν has a density relative to μ if and only if the following holds. For any $f : X \to \mathbb{R}$, $f \geq 0$, which is μ-integrable and ν-integrable, and for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if h is a function satisfying $0 \leq h \leq f$ and $\int^* h\, d\mu \leq \delta$, then $\int^* h\, d\nu \leq \varepsilon$.

Question 2. Let μ be a measure on X and $A \subset X$ be such that $X \backslash A$ is negligible. Prove that $\mu = \chi_A \mu$.

Question 3. Prove the $\mathcal{L}^p$–$\mathcal{L}^q$ duality stated in class, $p > 1$.

Question 4. In this problem you have to show that there exist continuous linear forms on $\mathcal{L}^\infty(X)$ that are not of the form $f \mapsto \int fg\, d\mu$ for some function $g \in \mathcal{L}^1(X)$. Proceed as follows. (If you know a different proof than what is outlined below you are welcome to present it.)

Take $X = [-1, 1]$ with μ to be the Lebesgue measure. Let $C^0(X)$ be the space of continuous real valued functions on X (we will take all functions to be real), and notice that $C^0(X) \subseteq L^\infty(X)$. Define a (linear) operator $\lambda : C^0(X) \to \mathbb{R}$ by $\lambda(f) = f(0)$. Show that this operator is continuous with respect to the topology induced on $C^0(X)$ from $\mathcal{L}^\infty(X)$. Invoking a theorem from last semester, conclude that λ extends to a continuous linear form on $\mathcal{L}^\infty(X)$. Suppose that λ could be written as $\lambda(f) = \int fg\, d\mu$ for a certain $g \in \mathcal{L}^1(X)$. Then, for any $f \in C^0(X)$ we have $\int fg\, d\mu = f(0)$. Show that for any $x \neq 0$ and $\varepsilon > 0$ such that $|x| > \varepsilon$, we can find a sequence of non-negative continuous functions $\{f_n\}_{n=1}^\infty$ supported on $[x-\varepsilon, x+\varepsilon]$ such that f_n converges in the $\mathcal{L}^\infty$-topology to $\chi_{[x-\varepsilon,x+\varepsilon]}$.(We can assume $[x-\varepsilon, x+\varepsilon] \subset [-1, 1]$ upon taking ε small.) Conclude that

$$\lambda(f_n) = \int f_n g\, d\mu = 0$$

for all n, and that

$$\int f_n g\, d\mu \to \int \chi_{[x-\varepsilon,x+\varepsilon]} g\, d\mu$$

as $n \to \infty$. Next, invoke the following theorem which will be proven later in the course:

Theorem 1. *(Lebesgue differentiation theorem) Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally integrable. Then for almost every $x \in \mathbb{R}^n$*

$$\lim_{r\to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} f\, d\mu \to f(x),$$

where $B_r(x)$ is the ball of radius r centered at x and $|B_r(x)|$ its volume.

Conclude that $g(x) = 0$ almost everywhere, thus $\lambda = 0$, which gives a contradiction.

Above, the continuous linear form λ defined on the whole of $\mathcal{L}^\infty(X)$ has an explicit formula when restricted to continuous function, but notice that $\lambda(f) = f(0)$ is not valid for arbitrary elements of $\mathcal{L}^\infty(X)$. Can you write an explicit formula for a continuous linear form on $\mathcal{L}^\infty(X)$ that is not given by integration against an $\mathcal{L}^1(X)$ function?

Question 5. Prove that two measures μ and ν are singular with respect to each other if and only if $\inf(|\mu|, |\nu|) = 0$.