REAL ANALYSIS, HW 3

VANDERBILT UNIVERSITY

supp	Support of a function or a measure
X	Locally compact (topological) space
K	Compact set in X
E	Locally convex (topological vector) space
$\mathscr{C}(X;E)$	Space of continuous functions from X to E endowed with the uniform topology
$\mathscr{C}_{c.o.}(X;E)$	Space of continuous functions from X to E endowed with the compact-open topology
$\mathscr{C}_c(X;E)$	Space of continuous functions from X to E with compact support endowed with the compact-open topology
$\mathscr{C}(K;E)$	Space of continuous functions from K to E endowed with the topology inherited from $\mathscr{C}(X,E)$
$\mathscr{K}(X;E)$	Space of continuous functions from X to E with compact support endowed with the inductive limit of locally convex topologies
$\mathscr{K}(X,A;E)$	Elements $f\in\mathscr{K}(X;E)$ such that $\operatorname{supp}(f)\subseteq A$
$\mathscr{K}(X,K;E)$	Elements $f\in\mathscr{K}(X;E)$ such that $\operatorname{supp}(f)\subseteq K$ endowed with the topology of compact convergence
$\mathscr{K}_+(X;\mathbb{R})$	Elements $f\in\mathscr{K}(X;\mathbb{R})$ such that $f\geq 0$
$\mathscr{K}(X)$	$\mathscr{K}(X;\mathbb{C})$ or $\mathscr{K}(X;\mathbb{R})$, with $\mathbb{C}$ or $\mathbb{R}$ understood from the context
$\mathscr{M}(X;\mathbb{C})$	Space of measures on X
$\mathscr{M}(X;\mathbb{R})$	Space of real measures on X
$\mathscr{M}_+(X;\mathbb{R})$	Space of positive measures on X
$\mathscr{I}_+(X;\mathbb{R})$	Space of positive (non-negative) lower semi-continuous functions on X
$\mu^*(f)$	Upper integral of f (with respect to the positive measure μ), also denoted $\int^* f\,d\mu$
χ_A	Characteristic function of the set A
$\mu^*(A)$	Outer measure of A (with respect to the positive measure μ)
$N_p(f)$	$(\lvert\mu\rvert^*(\lvert f\rvert^p))^{\frac{1}{p}}$, $1\leq p<\infty$
$\mathscr{F}^p(X)$	Maps f from X to $\mathbb{C}$ or $\mathbb{R}$ such that $N_p(f)<\infty$, with topology given by the semi-norm N_p. Depending on the context, $\mathscr{F}^p(X)$ can denote maps defined a.e. such that $N_p(f)<\infty$, and also taking values in $\overline{\mathbb{R}}$
$\mathscr{L}^p(X)$	Closure of $\mathscr{K}(X)$ in $\mathscr{F}^p(X)$
$L^p(X)$	Hausdorff space associated with $\mathscr{L}^p(X)$
$f\sim g$	Equivalence relation $f(x)=g(x)$ a.e.
$\widetilde{f}$	Equivalence class of f given by the equivalence relation $\sim$
$\mathscr{E}_F(\Phi)$	Set of Φ-step functions with values in F, where Φ is a Boolean ring and $F=\mathbb{R}$ or $\mathbb{C}$.

Recall that we also call the compact-open topology the topology of compact convergence. Unless stated otherwise, the ordering in the function spaces and spaces of measures is as defined in class and denoted $\leq$, when such relation is well-defined. Recall that by a set of zero measure we mean a set of zero outer measure. The topology on $\mathscr{F}^p(X)$ is called the topology of convergence of mean of order p, the L^p-topology, or yet the topology of convergence in L^p. Elements in $\mathscr{L}^p(X)$ are called p-integrable. This terminology is extended to functions defined a.e. and taking values in $\overline{\mathbb{R}}$ as done in class.

X will always denote a locally compact σ-compact space.

Question 1. Let μ and ν be positive measures on X. Prove that ν has a density relative to μ if and only if the following holds. For any $f : X \to \mathbb{R}$, $f \geq 0$, which is μ-integrable and ν-integrable, and for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if h is a function satisfying $0 \leq h \leq f$ and $\int^* h\, d\mu \leq \delta$, then $\int^* h\, d\nu \leq \varepsilon$.

Solution. Assume $\nu = g\mu$, where g is locally μ-integrable. Let us argue by contradiction. Suppose that there exist a function $f \geq 0$ that is μ and ν-integrable, and a real number $\varepsilon > 0$ such that, for any $n \in \mathbb{N}$, there exists a function g_n, $0 \leq g_n \leq f$, satisfying

$$\int^* g_n\, d\mu \leq 2^{-n}$$

and

$$\int^* g_n\, d\nu \geq \varepsilon.$$

From the definition of the upper integral we can replace g_n by $\inf(f, h_n)$ for some $h_n \in \mathscr{I}_+(X; \mathbb{R})$ without changing the previous inequalities, thus we can assume g_n to be μ and ν-integrable. Set

$$v_n = \sup_{\ell > 0} g_{n+\ell} \leq \sum_{\ell=1}^{\infty} g_{n+\ell}$$

and

$$v = \limsup_{n\to\infty} g_n = \inf_n v_n.$$

v_n is then μ and ν-integrable since $g_n \leq f$, thus an application of the dominated convergence theorem gives

$$\int v_n\, d\mu \leq \sum_{\ell=1}^{\infty} \int g_{n+\ell}\, d\mu \leq 2^{-n}.$$

Thus

$$\int v\, d\mu = 0$$

and the assumption $\nu = g\mu$ implies that v is ν-negligible. But

$$\int v\, d\nu = \lim_{n\to\infty} \int v_n\, d\nu \geq \varepsilon,$$

giving a contradiction.

Reciprocally, notice that the statement implies that every μ-negligible set is ν-negligible. Under these circumstances we proved in class that ν has a density with respect to μ.

Question 2. Let μ be a measure on X and $A \subset X$ be such that $X \backslash A$ is negligible. Prove that $\mu = \chi_A \mu$.

Solution. Notice that A is μ-measurable thus χ_A locally μ-integrable. Set $\nu = \chi_A \mu$. Then for any $f \in \mathscr{K}(X;\mathbb{R})$,

$$\int f \, d\nu = \int f \chi_A \, d\mu.$$

On the other hand, since A is μ-measurable,

$$\int f \, d\mu = \int_A f \, d\mu + \int_{A^c} f \, d\mu = \int f \chi_A \, d\mu,$$

establishing the result.

Question 3. Prove the $\mathscr{L}^p$–$\mathscr{L}^q$ duality stated in class, $p > 1$.

Solution. If $g \in \mathscr{L}^q$, then we obtain a continuous linear form in view of Hölder's inequality.

Now let λ be a (non-trivial) continuous linear form on $\mathscr{L}^p(X)$. Then $|\lambda(f)| \leq M N_p(f)$ for all $f \in \mathscr{L}^p(X)$ and some $M > 0$. Thus λ is continuous on $\mathscr{K}(X)$ and defines a measure ν.

Since $|\chi_A|^p = \chi_A$, for any μ-integrable $A \subseteq X$,

$$|\lambda(\chi_A)| \leq M N_p(\chi_A) = M \int \chi_A \, d\mu.$$

As λ and ν agree on $\mathscr{K}(X)$, we see that any μ-negligible set is also ν-negligible and thus $\nu = g\mu$ for some locally μ-integrable function g.

For any $f \in \mathscr{K}(X)$,

$$\lambda(f) = \nu(f) = \int f \, d\nu = \int f g \, d\mu.$$

Therefore, the linear forms $f \mapsto \lambda(f)$ and $f \mapsto \int fg \, d\mu$ agree on $\mathscr{K}(X)$, thus on $\mathscr{L}^p(X)$. It remains to show that $g \in \mathscr{L}^q(X)$.

As in the proof of the dual of $\mathscr{L}^1$, we can write $|g| = hg$ where $|h(x)| = 1$. Let $A_n = \{x \in X \mid |g(x)| \leq n\}$ and $f_n = \chi_{A_n} |g|^{q-1} h$. Then $|f_n|^p = |g|^q$ on A_n, and for any compact K,

$$\int_{A_n \cap K} |g|^q \, d\mu = \int \chi_K f_n g \, d\mu = \lambda(\chi_K f_n) \leq M N_p(\chi_K f_n) = M \left(\int_{A_n \cap K} |g|^q \, d\mu \right)^{\frac{1}{p}}.$$

From the monotone convergence theorem and σ-compactness, we obtain $N_q(g) \leq M$.

Question 4. In this problem you have to show that there exist continuous linear forms on $\mathscr{L}^\infty(X)$ that are not of the form $f \mapsto \int fg \, d\mu$ for some function $g \in \mathscr{L}^1(X)$. Proceed as follows. (If you know a different proof than what is outlined below you are welcome to present it.)

Take $X = [-1, 1]$ with μ to be the Lebesgue measure. Let $C^0(X)$ be the space of continuous real valued functions on X (we will take all functions to be real), and notice that $C^0(X) \subseteq L^\infty(X)$. Define a (linear) operator $\lambda : C^0(X) \to \mathbb{R}$ by $\lambda(f) = f(0)$. Show that this operator is continuous with respect to the topology induced on $C^0(X)$ from $\mathscr{L}^\infty(X)$. Invoking a theorem from last semester, conclude that λ extends to a continuous linear form on $\mathscr{L}^\infty(X)$. Suppose that λ could be written as $\lambda(f) = \int fg \, d\mu$ for a certain $g \in \mathscr{L}^1(X)$. Then, for any $f \in C^0(X)$ we have $\int fg \, d\mu = f(0)$. Show that for any $x \neq 0$ and $\varepsilon > 0$ such that $|x| > \varepsilon$, we can find a sequence of non-negative continuous functions $\{f_n\}_{n=1}^\infty$ supported on $[x-\varepsilon, x+\varepsilon]$ such that f_n converges in the $\mathscr{L}^\infty$-topology to $\chi_{[x-\varepsilon, x+\varepsilon]}$.(We can assume $[x-\varepsilon, x+\varepsilon] \subset [-1, 1]$ upon taking ε small.) Conclude that

$$\lambda(f_n) = \int f_n g \, d\mu = 0$$

for all n, and that

$$\int f_n g \, d\mu \to \int \chi_{[x-\varepsilon, x+\varepsilon]} g \, d\mu$$

as $n \to \infty$. Next, invoke the following theorem which will be proven later in the course:

Theorem 1. *(Lebesgue differentiation theorem) Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally integrable. Then for almost every $x \in \mathbb{R}^n$*

$$\lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} f \, d\mu \to f(x),$$

where $B_r(x)$ is the ball of radius r centered at x and $|B_r(x)|$ its volume.

Conclude that $g(x) = 0$ almost everywhere, thus $\lambda = 0$, which gives a contradiction.

Above, the continuous linear form λ defined on the whole of $\mathscr{L}^\infty(X)$ has an explicit formula when restricted to continuous function, but notice that $\lambda(f) = f(0)$ is not valid for arbitrary elements of $\mathscr{L}^\infty(X)$. Can you write an explicit formula for a continuous linear form on $\mathscr{L}^\infty(X)$ that is not given by integration against an $\mathscr{L}^1(X)$ function?

Solution. Notice that for any continuous function,

$$|\lambda(f)| = |f(0)| \leq N_\infty(f).$$

Hence λ defines a continuous linear form on the subspace $C^0(X)$ with respect to the $\mathscr{L}^\infty$-topology. By the Hahn-Banach theorem λ extends to a continuous linear form on the whole of $\mathscr{L}^\infty(X)$. Now we just follow the argument outlined in the statement of the exercise.

Recall that the Hanh-Banach was proven as a consequence of a separation theorem, which in turn relies on Zorn's lemma. As far as I know all proofs that there exist continuous linear forms on $\mathscr{L}^\infty(X)$ not given by integration rely on the Hahn-Banach or some similar proposition not provable in ZF, thus an explicit construction seems to be impossible.

Question 5. Prove that two measures μ and ν are singular with respect to each other if and only if $\inf(|\mu|, |\nu|) = 0$.

Solution. We can assume the measures to be positive, so $\mu = g\lambda$ and $\nu = h\lambda$ with $\lambda = \mu + \nu$. Then $\inf(\mu, \nu) = \inf(g, h)\lambda$. Thus, $\inf(\mu, \nu) = 0$ if and only if $\inf(g, h)$ is λ negligible. Letting M and N be the set of points where g and h do not vanish, respectively, $\inf(g, h)$ is λ-negligible if and only if $M \cap N$ is λ-negligible. Setting $M_1 = M \backslash (M \cap N)$ and $N_1 = N \backslash (M \cap N)$, the condition is then equivalent to $g = \chi_{M_1} g$ and $h = \chi_{N_1} h$ λ-almost everywhere. But $g = \chi_{M_1} g$ holds λ almost everywhere if and only if μ is concentrated on M_1; analogously for ν.