REAL ANALYSIS, HW 1

VANDERBILT UNIVERSITY

supp	Support of a function or a measure
X	Locally compact (topological) space
K	Compact set in X
E	Locally convex (topological vector) space
$\mathscr{C}(X;E)$	Space of continuous functions from X to E endowed with the uniform topology
$\mathscr{C}_{c.o.}(X;E)$	Space of continuous functions from X to E endowed with the compact-open topology
$\mathscr{C}_c(X;E)$	Space of continuous functions from X to E with compact support endowed with the compact-open topology
$\mathscr{C}(K;E)$	Space of continuous functions from K to E endowed with the topology inherited from $\mathscr{C}(X,E)$
$\mathscr{K}(X;E)$	Space of continuous functions from X to E with compact support endowed with the inductive limit of locally convex topologies
$\mathscr{K}(X,A;E)$	Elements $f \in \mathscr{K}(X;E)$ such that $\operatorname{supp}(f) \subseteq A$
$\mathscr{K}(X,K;E)$	Elements $f \in \mathscr{K}(X;E)$ such that $\operatorname{supp}(f) \subseteq K$ endowed with the topology of compact convergence
$\mathscr{K}_+(X;\mathbb{R})$	Elements $f \in \mathscr{K}(X;\mathbb{R})$ such that $f \geq 0$
$\mathscr{K}(X)$	$\mathscr{K}(X;\mathbb{C})$ or $\mathscr{K}(X;\mathbb{R})$, with $\mathbb{C}$ or $\mathbb{R}$ understood from the context
$\mathscr{M}(X;\mathbb{C})$	Space of measures on X
$\mathscr{M}(X;\mathbb{R})$	Space of real measures on X
$\mathscr{M}_+(X;\mathbb{R})$	Space of positive measures on X
$\mathscr{I}_+(X;\mathbb{R})$	Space of positive (non-negative) lower semi-continuous functions on X
$\mu^*(f)$	Upper integral of f (with respect to the positive measure μ), also denoted $\int^* f\,d\mu$
χ_A	Characteristic function of the set A
$\mu^*(A)$	Outer measure of A (with respect to the positive measure μ)
$N_p(f)$	$(\lvert\mu\rvert^*(\lvert f\rvert^p))^{\frac{1}{p}}$, $1 \leq p < \infty$
$\mathscr{F}^p(X)$	Maps f from X to $\mathbb{C}$ or $\mathbb{R}$ such that $N_p(f) < \infty$, with topology given by the semi-norm N_p. Depending on the context, $\mathscr{F}^p(X)$ can denote maps defined a.e. such that $N_p(f) < \infty$, and also taking values in $\overline{\mathbb{R}}$
$\mathscr{L}^p(X)$	Closure of $\mathscr{K}(X)$ in $\mathscr{F}^p(X)$
$L^p(X)$	Hausdorff space associated with $\mathscr{L}^p(X)$
$f \sim g$	Equivalence relation $f(x) = g(x)$ a.e.
$\widetilde{f}$	Equivalence class of f given by the equivalence relation $\sim$
$\mathscr{E}_F(\Phi)$	Set of Φ-step functions with values in F, where Φ is a Boolean ring and $F = \mathbb{R}$ or $\mathbb{C}$.

Recall that we also call the compact-open topology the topology of compact convergence. Unless stated otherwise, the ordering in the function spaces and spaces of measures is as defined in class and denoted $\leq$, when such relation is well-defined. Recall that by a set of zero measure we mean a set of zero outer measure. The topology on $\mathscr{F}^p(X)$ is called the topology of convergence of mean of order p, the L^p-topology, or yet the topology of convergence in L^p. Elements in $\mathscr{L}^p(X)$ are called p-integrable. This terminology is extended to functions defined a.e. and taking values in $\overline{\mathbb{R}}$ as done in class.

Below, X will always be a locally compact σ-compact space.

Question 1. Prove that $\mathscr{L}^\infty(X)$ is complete.

Solution. Since the topology on $\mathscr{L}^\infty(X)$ is generated by a single semi-norm, it suffices to consider Cauchy sequences. Let $\{f_n\}_{n=1}^\infty \subset \mathscr{L}^\infty(X)$ be a Cauchy sequence. Given $k \in \mathbb{N}$, we can find a N_k such that $N_\infty(f_m - f_n) \leq \frac{1}{k}$ for all $m, n \geq N_k$. For each $m, n \geq N_k$, set $A_{mnk} = \{x \in X \,|\, |f_m(x) - f_n(x)| > \frac{1}{k}\}$. Then A_{mnk} is negligible, and thus is their union A. It follows that $\{f_n(x)\}_{n=1}^\infty$ converges uniformly on $X \backslash A$; set $f(x)$ to be its limit (defined almost everywhere). f is then bounded on $X \backslash A$ and by Egoroff's theorem it is measurable; hence $f \in \mathscr{L}^\infty(X)$. Because $\{f_n\}_{n=1}^\infty$ converges uniformly to f on the complement on of a negligible set, we conclude (from the characterization of convergence in $\mathscr{L}^\infty(X)$) that $\{f_n\}_{n=1}^\infty$ converges to f in $\subset \mathscr{L}^\infty(X)$.

Question 2. If $1 \leq r < p < q \leq \infty$, then $\mathscr{L}^r \cap \mathscr{L}^q(X) \subset \mathscr{L}^p(x)$ and $N_p(f) \leq (N_r(f))^\theta (N_q(f))^{1-\theta}$, where $\frac{1}{p} = \frac{\theta}{r} + \frac{1-\theta}{q}$, $0 < \theta < 1$.

Solution. The cases where an exponent is ∞ follow from the mean value inequality. Otherwise, apply Hölder's inequality with $P = \frac{r}{\theta p}$ and $Q = \frac{q}{p(1-\theta)}$, since $\frac{1}{P} + \frac{1}{Q} = 1$.

Question 3. Let μ be a positive measure on X and $g \geq 0$ be a locally integrable function. Set $\nu = g\mu$. Then $f : X \to \overline{\mathbb{R}}$ is ν-integrable if and only if fg is μ-integrable, in which case

$$\int f \, d\nu = \int fg \, d\mu. \tag{1}$$

Solution. We proved in class that f is ν-measurable if and only if fg is μ-measurable. We also proved that for any numerical function $h \geq 0$,

$$\int^* h \, d\nu = \int^* hg \, d\mu.$$

Recall that for a function to be integrable it is necessary and sufficient that it be measurable and its upper integral be finite, hence the equivalence of integrability conditions. The equality of the integrals follows from $f = f^+ - f^-$.

Question 4. Let μ be a positive measure on X and $g \geq 0$ be a locally integrable function. Set $\nu = g\mu$. Then ν is bounded if and only if g is μ-integrable, and $\nu = 0$ if and only if g is μ-negligible.

Solution. The first statement follows at once from (1) applied to the constant function $f = 1$. For the second statement, if g is μ-negligible, then for any $f \in \mathscr{K}(X)$ we have, again from (1), $\nu(f) = \int fg \, d\mu = 0$, thus $\nu = 0$. Reciprocally, suppose that g is not negligible. Then, since $g \geq 0$, the (measurable) set $A = \{x \in X \,|\, g(x) > 0\}$ has positive μ-measure. Using that A is measurable and of positive measure, and that g is measurable, we see that we can find a compact set $K \subseteq A$ such that g is strictly positive on K. Let f be a non-negative continuous function with compact support that is identically equal to one on K (such function exists by a corollary of Urysohn's lemma). Then, from (1)

$$\int f \, d\nu \geq \int_K f \, d\nu = \int_K fg \, d\mu = \int_K g \, d\mu > 0,$$

thus $\nu \neq 0$.