REAL ANALYSIS, HW 1

VANDERBILT UNIVERSITY

supp	Support of a function or a measure
X	Locally compact (topological) space
K	Compact set in X
E	Locally convex (topological vector) space
$\mathscr{C}(X;E)$	Space of continuous functions from X to E endowed with the uniform topology
$\mathscr{C}_{c.o.}(X;E)$	Space of continuous functions from X to E endowed with the compact-open topology
$\mathscr{C}_c(X;E)$	Space of continuous functions from X to E with compact support endowed with the compact-open topology
$\mathscr{C}(K;E)$	Space of continuous functions from K to E endowed with the topology inherited from $\mathscr{C}(X,E)$
$\mathscr{K}(X;E)$	Space of continuous functions from X to E with compact support endowed with the inductive limit of locally convex topologies
$\mathscr{K}(X,A;E)$	Elements $f \in \mathscr{K}(X;E)$ such that $\mathrm{supp}(f) \subseteq A$
$\mathscr{K}(X,K;E)$	Elements $f \in \mathscr{K}(X;E)$ such that $\mathrm{supp}(f) \subseteq K$ endowed with the topology of compact convergence
$\mathscr{K}_+(X;\mathbb{R})$	Elements $f \in \mathscr{K}(X;\mathbb{R})$ such that $f \geq 0$
$\mathscr{K}(X)$	$\mathscr{K}(X;\mathbb{C})$ or $\mathscr{K}(X;\mathbb{R})$, with $\mathbb{C}$ or $\mathbb{R}$ understood from the context
$\mathscr{M}(X;\mathbb{C})$	Space of measures on X
$\mathscr{M}(X;\mathbb{R})$	Space of real measures on X
$\mathscr{M}_+(X;\mathbb{R})$	Space of positive measures on X
$\mathscr{I}_+(X;\mathbb{R})$	Space of positive (non-negative) lower semi-continuous functions on X
$\mu^*(f)$	Upper integral of f (with respect to the positive measure μ), also denoted $\int^* f\, d\mu$
χ_A	Characteristic function of the set A
$\mu^*(A)$	Outer measure of A (with respect to the positive measure μ)
$N_p(f)$	$(\lvert\mu\rvert^*(\lvert f\rvert^p))^{\frac{1}{p}}$, $1 \leq p < \infty$
$\mathscr{F}^p(X)$	Maps f from X to $\mathbb{C}$ or $\mathbb{R}$ such that $N_p(f) < \infty$, with topology given by the semi-norm N_p. Depending on the context, $\mathscr{F}^p(X)$ can denote maps defined a.e. such that $N_p(f) < \infty$, and also taking values in $\overline{\mathbb{R}}$
$\mathscr{L}^p(X)$	Closure of $\mathscr{K}(X)$ in $\mathscr{F}^p(X)$
$L^p(X)$	Hausdorff space associated with $\mathscr{L}^p(X)$
$f \sim g$	Equivalence relation $f(x) = g(x)$ a.e.
$\widetilde{f}$	Equivalence class of f given by the equivalence relation $\sim$
$\mathscr{E}_F(\Phi)$	Set of Φ-step functions with values in F, where Φ is a Boolean ring and $F = \mathbb{R}$ or $\mathbb{C}$.

Recall that we also call the compact-open topology the topology of compact convergence. Unless stated otherwise, the ordering in the function spaces and spaces of measures is as defined in class and denoted $\leq$, when such relation is well-defined. Recall that by a set of zero measure we mean a set of zero outer measure. The topology on $\mathscr{F}^p(X)$ is called the topology of convergence of mean of order p, the L^p-topology, or yet the topology of convergence in L^p. Elements in $\mathscr{L}^p(X)$ are called p-integrable. This terminology is extended to functions defined a.e. and taking values in $\overline{\mathbb{R}}$ as done in class.

Question 1. Prove the following properties of measurable sets: (i) every integrable set is measurable; (ii) the open and closed sets are measurable; (iii) the measurable sets for a σ-algebra.

Solution. Let us prove (i). Recall that we showed:

Claim 1. *A set $A \subseteq X$ is measurable if and only if $K \cap A$ is integrable for every compact K.*

Assume that A is integrable. Since any compact set is integrable and the (countable) intersection of integrable sets is integrable, we conclude that $A \cap K$ is integrable for any compact K, hence A is measurable.

Let us prove (ii). If A is closed and K is any compact set, then $A \cap K$ is compact, hence integrable. Thus A is measurable by claim 1. Next, let A be a measurable set. Then, for any compact K, $A^c \cap K = K \backslash (A \cap K)$. Since $A \cap K$ is integrable because A is measurable, we have that $A^c \cap K$ is the difference of two integrable sets, hence integrable, and we conclude that A^c is measurable by claim 1. In particular the open sets are measurable.

Let us prove (iii). Let $\{A_n\}_{n=1}^\infty$ be a sequence of measurable sets and set $A = \cap_n A_n$. For any compact K, $K \cap A_n$ is integrable by claim 1, thus $A \cap K$ is the countable intersection of integrable sets, hence integrable. We conclude, again by claim 1, that A is measurable. Since we showed that A^c is measurable as well, $\cup_n A_n$ is measurable.

Question 2. Prove that a set $A \subseteq X$ is negligible if and only if it is locally negligible and has finite outer measure. Prove that every locally negligible open set is negligible. Finally, show that every locally negligible set is negligible if the space X is σ-compact.

Solution. Let A be negligible, then clearly any $x \in A$ has a neighborhood that is negligible. Conversely, assume that A is locally negligible and $|\mu| * (A) < \infty$. Because A has finite outer measure, we can find an open set $U \supseteq A$ such that $|\mu|^*(U) < \infty$. Hence U is integrable, and thus there exist a negligible set $N \subseteq U$ and a sequence $\{K_n\}_{n=1}^\infty$ of compact sets such that $U = N \cup (\cup_n K_n)$. $A \cap N$ is negligible. Recall that we showed that

Claim 2. *A set $A \subseteq X$ is locally negligible if and only if $A \cap K$ is negligible for every compact set $K \subseteq X$.*

Thus $A \cap K_n$ is negligible for all n, and we conclude that A is the countable union of negligible sets, hence negligible.

To prove the second statement, let U be a locally negligible open set. By claim 2, $|\mu|(K) = 0$ for any compact set $K \subseteq U$. Recall that $|\mu|^*(U) = \{\sup |\mu|(K) \,|\, K \subseteq U,\, K \text{ compact }\}$, hence the result.

To prove the last statement, let $X = \cup_n K_n$, where the K_n's are compact. Let A be locally negligible. Since $A = \cup_n (A \cap K_n)$ and each $A \cap K_n$ is negligible by claim 2, we have that A, being the countable union of negligible sets, is negligible.

Question 3. Let $f : X \to Y$ be measurable, Y a topological space. If $g : X \to Y$ is locally almost everywhere equal to f, then g is measurable.

Solution. Let N be the set of points $x \in X$ such that $f(x) \neq g(x)$, which is locally negligible by hypothesis. For any $x \in X$, let $K_x \ni x$ be a compact (hence integrable) neighborhood of x (which exists by local compactness). Then $K_x \cap N$ is negligible by claim 2. If we define $f_x = f$, we have that $g = f_x$ almost everywhere in K_x, and the result follows from the following claim proved in class:

Claim 3. *Let $f : X \to Y$, Y a topological space. Assume that for every $x \in X$ there exist an integrable neighborhood $V_x \ni x$ and a measurable function $g_x : X \to Y$ such that $f(y) = g_x(y)$ almost everywhere in V_x. Then f is measurable.*

Question 4. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable numerical functions. Prove that $\sup_n f_n$, $\inf_n f_n$, $\limsup_{n\to\infty} f_n$, and $\liminf_{n\to\infty} f_n$ are measurable.

Solution. Recall that the sup of finitely many measurable functions is measurable. Hence $g_n = \sup(f_1, \dots, f_n)$ is measurable, and $\sup_n f_n(x) = \lim_{n\to\infty} g_n(x)$. Hence $\sup_n f$ is measurable by Egoroff's theorem. This then implies that $h_n = \sup_\ell f_{n+\ell}$ is measurable, hence $\limsup_{n\to\infty} f_n$ is measurable by Egoroff's theorem in that $\limsup_{n\to\infty} f_n(x) = \lim_{n\to\infty} h_n(x)$. (Notice that the pointwise limits of g and h exist because these functions are increasing and decreasing, respectively). The inf statements are similar.

Question 5. Let $f : X \to Y$, Y a metric space with a countable basis. Prove that f is measurable if and only if $f^{-1}(\overline{B_r(p)})$ is measurable for every $r \in \mathbb{Q}$ and every p belonging to a dense countable subset of Y. Conclude that a numerical function is measurable if and only if $\{x \in X \,|\, f(x) \geq q\}$ is measurable for every $q \in \mathbb{Q}$.

Solution. Notice that Y has a countable dense subset D: pick a point in each basis element.

If f is measurable then $f^{-1}(\overline{B_r(p)})$ is measurable because the inverse image of a closed set under a measurable function is measurable (discussed in class).

To prove the conserve, enumerate the set D and write $D = \cup_{n=1}^{\infty}\{y_n\}$. Given a compact set K, let $A_{n,m}(K) = \{x \in K \,|\, d(f(x), y_n) \leq \frac{1}{m}\}$, $m \in \mathbb{N}$, where d is the metric on Y. In other words, $A_{n,m}(K) = f^{-1}(\overline{B_{\frac{1}{m}}(y_n)}) \cap K$. Since $f^{-1}(\overline{B_{\frac{1}{m}}(y_n)})$ is measurable by assumption and K is integrble, hence measurable, we have that $A_{n,m}(K)$ is measurable (σ-algebra property). We have $K = \cup_{n=1}^{\infty} A_{n,m}(K)$ for each fixed m. For, given $m \in \mathbb{N}$ and $x \in K$, there exists a sequence of points in D converging to $f(x)$ since D is dense in Y. Thus, there exists a $y_n \in D$ such that $d(f(x), y_n) \leq \frac{1}{m}$, and we have $x \in A_{n,m}(K)$. The reverse inclusion holds because $\cup_n \overline{B_{\frac{1}{m}}(y_n)} = Y$.

Define, inductively, $A'_{n+1,m}(K) = A_{n+1,m}(K) \backslash (\cup_{\ell=1}^{n} A_{\ell,m}(K))$, with $A'_{1,m}(K) = A_{1,m}(K)$. Then the sets $A'_{n+1,m}(K)$ are disjoint (it is useful to draw a picture to see this) and their union equals K since $K = \cup_{n=1}^{\infty} A_{n,m}(K)$, as noted above. Also, $A'_{n+1,m}(K)$ is measurable since it can equivalently be written as $A'_{n+1,m}(K) = A_{n+1,m}(K) \cap (\cup_{\ell \leq n} A_{\ell,m}(K))^c$. (Some of the $A'_{n+1,m}(K)$ might be empty, but at least $A'_{1,m}(K)$ is not).

For each $k \in \mathbb{N}$, define $B_{k,m}(K) = \cup_{n=1}^{k} A'_{k,m}(K)$. Then $B_{k,m}(K)$ is measurable. $B_{k,m}(K)$ is never empty since at least $A'_{1,m}(K) \neq \varnothing$. We have that $K = \cup_{k=1}^{\infty} A'_{k,m}(K) = B_{k,m}(K) \cup \cup_{n=k+1}^{\infty} A'_{n,m}(K)$ for each fixed m.

Fix $z \in Y$ and define $f_{k,m} : X \to Y$ by

$$f_{k,m}(x) = \begin{cases} z & \text{if } x \in X\backslash B_{k,m}(K), \\ y_n & \text{if } x \in A'_{n,m}(K) \text{ for some (unique by disjointness) } n \in \{1, \dots, k\}. \end{cases}$$

It is useful to note that $X\backslash B_{k,m}(K) = (X\backslash K) \cup \cup_{n=k+1}^{\infty} A'_{n,m}(K)$. Notice that $f_{k,m}$ is a measurable step function, hence measurable.

Let us investigate the limit, as $k \to \infty$, of $f_{k,m}$. If $x \in X\backslash K$, then $f_{k,m}(x) = z$ for every k, so $\lim_{k\to\infty} f_{k,m}(x) = z$. If $x \in K$, then $x \in A'_{n,m}(K)$ for a unique n, and $f_{k,m}(x) = f_{n,m}(x) = y_n$ for all $k \geq n$, thus $\lim_{k\to\infty} f_{k,m}(x) = y_n$. Therefore, if we define $f_m : X \to Y$ by

$$f_m(x) = \begin{cases} z & \text{if } x \in X\backslash K, \\ y_n & \text{if } x \in A'_{n,m}(K) \text{ for some (unique) } n, \end{cases}$$

we obtain that $f_{k,m}$ converges pointwise to f_m. By Egoroff's theorem, f_m is measurable.

Next, we want to compute the limit $\lim_{m\to\infty} f_m$. Let $x \in K$. Then, for any $m \in \mathbb{N}$, $x \in A'_{n,m}(K)$ for some n. If $k \geq n$, we have that $f_{k,m}(x) = f_{n,m}(x) = y_n$. Hence, by construction of $A'_{n,m}(K)$ and definition of $A_{n,m}(K)$, we have $\frac{1}{m} \geq d(f(x), y_n) = d(f(x), f_{n,m}(x)) = d(f(x), f_{k,m}(x))$, $k \geq n$. Since f_m is the limit as $k \to \infty$ of $f_{k,m}$, we have $d(f(x), f_m(x)) \leq \frac{1}{m}$, thus $f_m(x) \to f(x)$ as $m \to \infty$. If $x \in X\backslash K$, then $f_m(x) = z$ for all $m \in \mathbb{N}$. We conclude that f_m converges pointwise to the function $f_K : X \to Y$ given by

$$f_K(x) = \begin{cases} z & \text{if } x \in X\backslash K, \\ f(x) & \text{if } x \in K. \end{cases}$$

Since f_m is measurable, so is f_K by Egoroff's theorem. Invoking claim 3 and the fact that X is locally compact, we conclude that f is measurable.

Let us move to the second statement, focusing on sufficiency. So assume that f is a numerical function and that $\{x \in X \,|\, f(x) \geq q\}$ is measurable for every $q \in \mathbb{Q}$. Given $y \in \overline{\mathbb{R}}$, we have $\{x \in X \,|\, f(x) \geq y\} = \cap_{q\in\mathbb{Q}, q\leq y}\{x \in X \,|\, f(x) \geq q\}$, which is a countable intersection of measurable sets, hence measurable. By the σ-algebra property of measurable sets, $\{x \in X \,|\, f(x) < y\}$ is measurable. If $y \neq \pm\infty$, then $\{x \in X \,|\, f(x) \leq y\} = \cap_{m\geq 1}\{x \in X \,|\, f(x) < y + \frac{1}{m}\}$, which is a countable intersection of measurable sets, thus measurable. If $y = -\infty$, then $\{x \in X \,|\, f(x) \leq -\infty\} = \cap_{m\in\mathbb{Z}}\{x \in X \,|\, f(x) < n\}$, which is again a countable intersection of measurable sets. Finally, if $b = +\infty$, $\{x \in X \,|\, f(x) \leq +\infty\} = X$, hence measurable. We conclude that $\{x \in X \,|\, f(x) \leq y\}$ and $\{x \in X \,|\, f(x) \geq y\}$ are both measurable for any $y \in \overline{\mathbb{R}}$. In order to invoke the first part of the problem, we need to recall the structure of the closed balls in $\overline{\mathbb{R}}$. The topology on $\overline{\mathbb{R}}$ is generated by the open intervals on $\overline{\mathbb{R}}$. It follows that $\overline{\mathbb{R}}$ is homeomorphic to $[-1, 1]$. If we define $z(x) = x/(1+|x|)$ for $x \neq \pm\infty$, $z(-\infty) = -1$ and $z(+\infty) = 1$, then z is an order-preserving homeomorphism of $\overline{\mathbb{R}}$ onto $[-1, 1]$. A metric on $\overline{\mathbb{R}}$ compatible with its topology is given by $d(x, y) = |z(x) - z(y)|$. Thus the closed balls in $\overline{\mathbb{R}}$ are of the form $z^{-1}(I)$ where I is a closed sub-interval of $[-1, 1]$. From the previous arguments we then conclude that the closed balls in $\overline{\mathbb{R}}$ are the intersection of two measurable sets, hence measurable. f is thus measurable by the previous result. The other direction is immediate.