In class, we proved the following:

Theorem 1. *Let E be a topological vector space, $U \neq \varnothing$, a convex open subset of E, V a linear affine subvariety of E such that $V \cap U = \varnothing$. Then there exists a closed hyperplane containing V and not meeting U.*

We want to use this result (sometimes called the geometric Hahn-Banach theorem) to prove the following:

Theorem 2. *(Hahn-Banach) Let E be a topological vector space, p a continuous semi-norm on E, V a vector sub-space of E, and x' a linear form on V of norm ≤ 1 (for the semi-norm induced by p). Then x' is the restriction of a (continuous) linear form on E of norm ≤ 1.*

Proof. Begin noticing that since x' is bounded by p and p is continuous, x' is continuous. Next, observe that there exists a one-to-one correspondence between closed non-homogeneous hyperplanes in E and continuous linear forms on E. Each closed non-homogeneous hyperplance H in E can be uniquely identified with the continuous linear form y' such that $H = \{x \in E \,|\, \langle y', x \rangle = 1\}$.

Define $W = \{x \in V \,|\, \langle x', x \rangle = 1\}$, so that W is a closed hyperplane in V. Set $U = \{x \in E \,|\, p(x) < 1\}$. One sees that U is open and convex. Since by assumption the linear form x' has norm less than one, it follows that $W \cap U = \varnothing$. By the above theorem, there exists a closed hyperplane H in E containing W and not intersecting U, and by the previous remarks H is given by $H = \{x \in E \,|\, \langle y', x \rangle = 1\}$ for some continuous linear form y'. We have $|\langle y', x \rangle| < 1$ for $x \in U$ since H does not contain U, hence the norm of y' is ≤ 1.

Consider now $H' = \{x \in V \,|\, \langle y', x \rangle = 1\}$, i.e., $H \cap V$. H' is a hyperplane in V that contains W, thus it equals W. In view of the aforementioned one-to-one correspondence, it follows that $y' = x'$ in V.

$\square$