REAL ANALYSIS, HW 9

VANDERBILT UNIVERSITY

supp	Support of a function or a measure
X	Locally compact (topological) space
K	Compact set in X
E	Locally convex (topological vector) space
$\mathscr{C}(X;E)$	Space of continuous functions from X to E endowed with the uniform topology
$\mathscr{C}_{c.o.}(X;E)$	Space of continuous functions from X to E endowed with the compact-open topology
$\mathscr{C}_c(X;E)$	Space of continuous functions from X to E with compact support endowed with the compact-open topology
$\mathscr{C}(K;E)$	Space of continuous functions from K to E endowed with the topology inherited from $\mathscr{C}(X,E)$
$\mathscr{K}(X;E)$	Space of continuous functions from X to E with compact support endowed with the inductive limit of locally convex topologies
$\mathscr{K}(X,A;E)$	Elements $f \in \mathscr{K}(X;E)$ such that $\text{supp}(f) \subseteq A$
$\mathscr{K}(X,K;E)$	Elements $f \in \mathscr{K}(X;E)$ such that $\text{supp}(f) \subseteq K$ endowed with the topology of compact convergence
$\mathscr{K}_+(X;\mathbb{R})$	Elements $f \in \mathscr{K}(X;\mathbb{R})$ such that $f \geq 0$
$\mathscr{K}(X)$	$\mathscr{K}(X;\mathbb{C})$ or $\mathscr{K}(X;\mathbb{R})$, with $\mathbb{C}$ or $\mathbb{R}$ understood from the context
$\mathscr{M}(X;\mathbb{C})$	Space of measures on X
$\mathscr{M}(X;\mathbb{R})$	Space of real measures on X
$\mathscr{M}_+(X;\mathbb{R})$	Space of positive measures on X
$\mathscr{I}_+(X;\mathbb{R})$	Space of positive (non-negative) lower semi-continuous functions on X
$\mu^*(f)$	Upper integral of f (with respect to the positive measure μ), also denoted $\int^* f\, d\mu$
χ_A	Characteristic function of the set A
$\mu^*(A)$	Outer measure of A (with respect to the positive measure μ)
$N_p(f)$	$(\lvert\mu\rvert^*(\lvert f\rvert^p))^{\frac{1}{p}}$, $1 \leq p < \infty$
$\mathscr{F}^p(X)$	Maps f from X to $\mathbb{C}$ or $\mathbb{R}$ such that $N_p(f) < \infty$, with topology given by the semi-norm N_p. Depending on the context, $\mathscr{F}^p(X)$ can denote maps defined a.e. such that $N_p(f) < \infty$, and also taking values in $\overline{\mathbb{R}}$
$\mathscr{L}^p(X)$	Closure of $\mathscr{K}(X)$ in $\mathscr{F}^p(X)$
$L^p(X)$	Hausdorff space associated with $\mathscr{L}^p(X)$
$f \sim g$	Equivalence relation $f(x) = g(x)$ a.e.
$\widetilde{f}$	Equivalence class of f given by the equivalence relation $\sim$
$\mathscr{E}_F(\Phi)$	Set of Φ-step functions with values in F, where Φ is a Boolean ring and $F = \mathbb{R}$ or $\mathbb{C}$.

Recall that we also call the compact-open topology the topology of compact convergence. Unless stated otherwise, the ordering in the function spaces and spaces of measures is as defined in class and denoted $\leq$, when such relation is well-defined. Recall that by a set of zero measure we mean a set of zero outer measure. The topology on $\mathscr{F}^p(X)$ is called the topology of convergence of mean of order p, the L^p-topology, or yet the topology of convergence in L^p. Elements in $\mathscr{L}^p(X)$ are called p-integrable. This terminology is extended to functions defined a.e. and taking values in $\overline{\mathbb{R}}$ as done in class.

Question 1. Let $f \in \mathscr{L}^p(X)$ and A be an integrable set. Show that $f\chi_A \in \mathscr{L}^p(X)$. In particular, if f and A are integrable so is $f\chi_A$, in which case we define the integral of f **over the set** A as

$$\int_A f\, d\mu = \int f\chi_A\, d\mu.$$

Question 2. Let μ be the Lebesgue measure on $\mathbb{R}$. Show that every step function with compact support is integrable. Let f be a compactly supported function that is the uniform limit of a sequence of step functions $\{f_n\}_{n=1}^{\infty}$. Show that f is integrable and $\int f\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu$. Moreover, $\int f\, d\mu$ equals the improper Riemann integral $\int_{-\infty}^{\infty} f(x)\, dx$.

Question 3. Let μ be the Lebesgue measure on $\mathbb{R}$. Let f be an integrable function that is the uniform limit of a sequence of step functions. Let $I_n = [-n, n]$. Show that $\int |f|\, d\mu = \lim_{n\to\infty} \int |f|\chi_{I_n}\, d\mu = \lim_{n\to\infty} \int_{-n}^{n} |f(x)|\, dx$. Conclude that $\int_{-\infty}^{\infty} f(x)\, dx$ is absolutely convergent and that $\int f\, d\mu = \int_{-\infty}^{\infty} f(x)\, dx$.

Question 4. Let f be integrable, A an integrable open set and $\mu|_A$ the restriction of the measure μ from X to A. Is it true that

$$\int_A f\, d\mu = \int f|_A\, d\mu|_A \ ?$$

Question 5. In class we defined a σ-algebra as a Boolean algebra Φ that is closed under countable unions. Show that this is equivalent to the conditions (i) $\varnothing \in \Phi$; (ii) $M^c \in \Phi$ if $M \in \Phi$; (iii) $\cup_{i=1}^{\infty} M_i \in \Phi$ if each $M_i \in \Phi$.

Question 6. Show that $\mathscr{E}_F(\Phi)$ is dense in $\mathscr{L}^p(X)$

Question 7. Let μ and ν be two measures on X such that $\mu(K) = \nu(K)$ for every compact K. Conclude that $\mu = \nu$.

Question 8. In the proof of the Riesz representation theorem, we introduced a topology on 2^X. Is such topology Hausdorff?

Question 9. Establish the claims left as exercise in the proof of the Riesz representation theorem.

Question 10. Establish the existence of the Lebesgue measure via the "geometric construction" indicated in class.