REAL ANALYSIS, HW 4

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The notation used below follows the one used in class and should be self-explanatory. Directions similar to those of previous homework assignments continue to hold, in particular (i) sometimes a definition that has not been given in class is used in an exercises. It is expected that students will be able to figure out the obvious interpretation, but you can always consult the literature if necessary; (ii) problems are written in an understandable, but loose fashion. When necessary or convenient, first make a precise statement of what is being asked before presenting your solution.

Unless stated otherwise, the following notation is adopted throughout (which is the same used in class):

supp	Support of a function
X	Locally compact (topological) space
K	Compact set in X
E	Locally convex (topological vector) space
$\mathscr{C}(X;E)$	Space of continuous functions from X to E endowed with the compact-open topology
$\mathscr{C}(K;E)$	Space of continuous functions from K to E endowed with the topology inherited from $\mathscr{C}(X,E)$
$\mathscr{K}(X;E)$	Space of continuous functions from X to E with compact support endowed with the inductive limit of locally convex topologies
$\mathscr{K}(X,A;E)$	Elements $f \in \mathscr{K}(X;E)$ such that $\mathrm{supp}(f) \subseteq A$

Recall that we also call the compact-open topology the topology of compact convergence.

Question 1. Prove that $\mathscr{K}(X;E)$ is Hausdorff if E is. Next, show that the topology on $\mathscr{K}(X,K;E)$ induced by $\mathscr{K}(X;E)$ is the topology of compact convergence[1] in K, and that each $\mathscr{K}(X,K;E)$ is closed in $\mathscr{K}(X;E)$.

Question 2. We saw in class that if $T : E \to F$ is a continuous linear map between two topological vector spaces, then T is bounded. Show that the converse holds if the topologies of E and F are generated by a norm.

Question 3. Let (Y,d) be a metric space. Recall that for metric spaces we defined a topology called the topology of compact convergence with respect to the metric, with neighborhoods given by

$$B(f,K,\epsilon) = \{g \in \mathscr{C}(X,Y) \mid \sup_{x\in K} d(f(x),g(x)) < \epsilon\}.$$

Assume that the topology of E is generated by a norm, so in particular E is a metric space. Show that the compact-open topology and the topology of compact convergence with respect to the metric given by the norm are equivalent.

[1] Which in this case agrees with topology of uniform convergence which was mentioned but not defined in class.

Question 4. Let $\Phi = \{\varphi_\alpha\}_{\alpha \in A}$ be a family of continuous maps from X to E such that $\text{supp}(\varphi_\alpha) \subset K$ for all $\alpha \in A$, where K is a fixed compact set. Let p be a continuous semi-norm on E. Assume that Φ satisfies the following property[2]. Given $\epsilon > 0$ and a compact set K' such that $K' \subset U \subset K$, where U is open, there exists a finite open covering $\{V_i\}_{i=1}^k$ of K' by open sets of X, with $V_i \subset K$, $i = 1, \dots, k$, such that for all $\alpha \in A$, whenever $x, y \in V_i$ for some $i \in \{1, \dots, k\}$, one has $p(\varphi_\alpha(x) - \varphi_\alpha(y)) < \varepsilon$. Prove the following. Given ε, there exists a finite collection of open sets $\{U_i\}_{i=1}^n$ and a partition of the unity $\{\psi_i\}_{i=1}^n$ subordinate to $\{U_i\}_{i=1}^n$, such that $\text{supp}(\psi_i) \subset K$ for all $i = 1, \dots, n$, and if x_i is a point belonging to $\text{supp}(\psi_i)$ for some $i \in \{1, \dots, n\}$, then, for every $x \in X$ and every $\alpha \in A$, it holds that

$$p(\varphi_\alpha(x) - \sum_{i=1}^n \psi_i(x)\varphi_\alpha(x_i)) < \varepsilon.$$

Question 5. Assume that X is compact, and E is normed. Show that

$$\| f \| = \sup_{x \in X} \| f(x) \|_E,$$

where $\| \cdot \|_E$ is the norm on E and $f \in \mathscr{C}(X; E)$, defines a norm.

Question 6. Recall that the inclusion $K \subset X$ induces a map $i_K : \mathscr{K}(X, K; E) \to \mathscr{K}(X; E)$ which is continuous by the definition of the topology on $\mathscr{K}(X; E)$. Let F be a locally convex space and $f : \mathscr{K}(X; E) \to F$ a linear map. Prove that f is continuous if and only if each $f \circ i_K$ is continuous.

Question 7. Assume that the topology of E is given by a norm $\| \cdot \|_E$ and let F by a normed vector space with norm $\| \cdot \|_F$. Let $\mu : \mathscr{K}(X; \mathbb{C}) \to F$ be a linear map. Without using the general theorem proved in class that characterizes the continuity of maps from $\mathscr{K}(X; \mathbb{C})$ to F in terms of families of compact sets in X, prove that μ is continuous if and only if the following holds. For each $K \subseteq X$, there exists a constant M_K such that for every $f \in \mathscr{K}(X; E)$ with $\text{supp}(f) \subseteq K$, we have

$$\| \mu(f) \|_F \leq M_K \| f \|,$$

where $\| f \|$ is given by

$$\| f \| = \sup_{x \in X} \| f(x) \|_E .$$

Conclude that a linear form μ on $\mathscr{K}(X; \mathbb{C})$ is a measure if and only if for each $K \subseteq X$, there exists a constant M_K such that for every $f \in \mathscr{K}(X; E)$ with $\text{supp}(f) \subseteq K$, we have

$$|\mu(f)| \leq M_K \sup_{x \in X} |f(x)|$$

where $| \cdot |$ is the absolute value in $\mathbb{C}$.

The next three questions deal with properties of finite-dimensional topological vector spaces. You are free to use any result of linear algebra of finite dimensional spaces that you want, but make sure to state the theorems that you are using.

Question 8. Prove that all norms in $\mathbb{R}^n$ are equivalent. Conclude, invoking a previous HW problem, that all topologies given by a norm in $\mathbb{R}^n$ are equivalent.

Question 9. A map between topological vector spaces is called a **topological isomorphism** if it is an isomorphism of vector spaces which is also a **homeomorphism** (i.e., a continuous bijective

[2] The stated property is satisfied if Φ is equicontinuous, a concept that will be defined later on in the course.

map whose inverse is also continuous). Let E be a finite dimensional (Hausdorff) topological vector space (over $\mathbb{R}$). Prove that it is topologically isomorphic to $\mathbb{R}^n$ for some n.

Question 10. Prove that every linear map between finite dimensional topological vector spaces is continuous.