REAL ANALYSIS, HW 2

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The notation used below follows the one used in class and should be self-explanatory. Sometimes a definition that has not been given in class is used in an exercises, such as "bounded metric" in exercise 3. It is expected that students will be able to figure out the obvious interpretation, but you can always consult the literature if necessary. The problems are written in an understandable, but loose fashion. When necessary or convenient, first make a precise statement of what is being asked before presenting your solution. For instance, problem 5 should read "Let (X, τ), (X', τ'), and (X'', τ'') be topological spaces, $f : X \to X'$ and $g : X' \to X''$ be continuous functions such that..."

Question 1. Show that the following defines a norm:

(a) $\| f \|_1 = \int_0^1 |f(x)|\, dx,$ for f a continuous real valued function on $[0, 1]$.

(b) $\| f \|_\infty = \sup_{x\in[0,1]} |f(x)|,$ for f a continuous real valued function on $[0, 1]$.

Question 2. Let α be a real number $0 < \alpha \leq 1$. A real valued function on $[0, 1]$ is called Hölder continuous of order α if there exists a constant C such that for all x, y we have

$$|f(x) - f(y)| \leq C|x - y|^\alpha \tag{1}$$

For such a function, define

$$\| f \|_\alpha = \sup_x |f(x)| + \sup_{\substack{x,y \\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}. \tag{2}$$

Show that the set of functions satisfying (1) forms a vector space, and that (2) defines a norm in this space.

Question 3. If d is a metric on X, show that

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is a bounded metric (definition?) that gives the same topology of X.

Question 4. Show that the example given in class is in fact a countable basis for the topology of $\mathbb{R}^n$ (if you did not come to class or did not take notes, the problem reads: find a countable basis for the ordinary topology of $\mathbb{R}^n$).

Question 5. Show that the composition of continuous functions is continuous.

Question 6. Show that two norms $|\cdot|_1$ and $|\cdot|_2$ on a vector space E give rise to the same topology if and only if there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1|x|_1 \leq |x|_2 \leq C_2|x|_1,$$

for all $x \in E$.

Question 7. Prove that a topological space is compact if and only if for any family of closed sets having the finite intersection property, the family has non-empty intersection. Remark: we proved one direction in class, you only need to prove the converse.

Question 8. Let A be a subset of a topological space, and assume that for each $x \in A$ there exists an open set U such that $x \in U \subset A$. Show that A is open.

Question 9. Give the definition of a topological vector space and explain what is meant by saying that the vector space operations are continuous (what are the topologies involved?).

Question 10. Suppose that K and A are subsets of a topological vector space E. Assume that K is compact, A is closed, and $K \cap A = \varnothing$. Then 0 has a neighborhood V such that

$$(K + V) \cap (A + V) = \varnothing.$$

Use this result to conclude that every topological vector space is a Hausdorff space (here we are using the definition of a topological vector space where points are closed; the result is not true otherwise).