VANDERBILT UNIVERSITY

MATH 4110 – PARTIAL DIFFERENTIAL EQUATIONS

Practice problems for HW 6

In the questions below, we follow the notation employed in class.

Question 1. For each set Ω below: (i) describe Ω in words (e.g., the first quadrant, intersection of the ball of radius one with the third quadrant, etc), drawing a picture when possible; (ii) identify $\partial\Omega$, and $\overline{\Omega}$; (iii) identify the area element of the boundary, i.e., dS; (iv) identify the normal to the boundary. The notation $B_r(z)$ is used for the (open) ball of radius r centered at $z \in \mathbb{R}^n$.

(a)

$$\Omega = \left\{ x \in \mathbb{R}^2 \,\middle|\, |x| < 5 \right\}.$$

(b)

$$\Omega = \left\{ x \in \mathbb{R}^2 \,\middle|\, -2 < x_1 < 2, -1 < x_2 < 1 \right\}.$$

(c)

$$\Omega = \left\{ x \in \mathbb{R}^3 \,\middle|\, -1 < x_1 < 1, -1 < x_2 < 1, -1 < x_3 < 1 \right\}.$$

(d)

$$\Omega = \left\{ x \in \mathbb{R}^2 \,\middle|\, -1 < x_1 < 1, -1 < x_2 < 1 \right\} \bigcap B_1(0).$$

(e)

$$\Omega = \left\{ x \in \mathbb{R}^3 \,\middle|\, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \right\}.$$

(f)

$$\Omega = \left\{ x \in \mathbb{R}^3 \,\middle|\, x_3 > 0 \right\} \bigcap B_r(0).$$

(g)

$$\Omega = \left\{ x \in \mathbb{R}^3 \,\middle|\, \langle x, (1,1,1) \rangle = 0 \right\} \bigcap B_1(0).$$

Question 2. Recall the integration by parts formula in several dimensions:

$$\int_\Omega f \frac{\partial g}{\partial x_i} = -\int_\Omega \frac{\partial f}{\partial x_i} g + \int_{\partial\Omega} f g \nu_i. \tag{1}$$

Use (1) to prove the following formulas:

(a) Green's first identity:

$$\int_\Omega \nabla f \cdot \nabla g = -\int_\Omega f \Delta g + \int_{\partial\Omega} f \nabla g \cdot \nu.$$

(b) Green's second identity:

$$\int_\Omega (f \Delta g - g \Delta f) = \int_{\partial\Omega} (f \nabla g \cdot \nu - g \nabla f \cdot \nu).$$

(c)

$$\int_{\Omega} \Delta f = \int_{\partial\Omega} \nabla f \cdot \nu.$$

(d) Divergence theorem:

$$\int_{\Omega} \operatorname{div} F = \int_{\partial\Omega} F \cdot \nu.$$

Question 3. Show that Laplace's equation is invariant under rotations, i.e., let u solve $\Delta u = 0$ and let M be an orthogonal matrix. Set

$$\widetilde{u}(x) = u(Mx).$$

Show that $\Delta \widetilde{u} = 0$.

The next questions deal with some more advanced notions and provide some insight into some mathematical concepts underlying the course content, but that have not been thoroughly emphasized in class. They require concepts that you you learned in previous courses (such as vector spaces, etc). These problems are suggested only if you want to go a bit beyond the "mechanics" of solving PDEs.

In all questions below, let Ω be a domain in $\mathbb{R}^n$, i.e., Ω is an open and connected set contained in $\mathbb{R}^n$. For concreteness you can imagine that Ω is the ball of radius one centered at the origin.

Recall that we say that a function u is k-times continuously differentiable if all derivatives up to order k of u exist and are continuous.

Remember that we defined the spaces

$$C^k(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \,\Big|\, u \text{ is } k\text{-times continuously differentiable } \Big\}.$$

Question 4. To get a better understanding of the space $C^k(\Omega)$, consider the following function:

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Show that (a) f is continuous, (b) f is differentiable, i.e., $f'(x)$ exists for all $x \in \mathbb{R}$, but (c) f is not C^1, i.e., $f'(x)$ is not continuous. That is why in the definition of C^k we require not only that the derivatives exists, but also that they are continuous.
Hint: You should focus on what happens for $x = 0$.

Question 5. Show that $C^k(\Omega)$ is a vector space.

Question 6. Show that the Laplacian Δ is a linear map between $C^k(\Omega)$ and $C^{k-2}(\Omega)$, $k \geq 2$.

Question 7. Recall that

$$C^\infty(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \,\Big|\, u \in C^k(\Omega) \text{ for every } k \Big\}.$$

Show that $C^\infty(\Omega)$ is a vector space and that the Laplacian Δ is a linear map from $C^\infty(\Omega)$ to itself.

Question 8. Give a reasonable argument for why $C^k(\Omega)$ is an infinite-dimensional vector space. You are not asked to provide a mathematical and rigorous proof. Instead, you should use your knowledge of calculus and linear algebra, as well as the material we learned in class, to construct a sensible explanation, even if only an intuitive one.